Kaluza-Klein reduction of generalised theories of gravity and non-minimal gauge couplings

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Abstract. The Kaluza-Klein reduction of a generalised theory of gravity in $D = 5$ dimensions is given. The form of the interactions among the gravitational, electromagnetic and massless scalar fields in four dimensional spacetime is exhibited.

1. Introduction

The gravitational fields in four-dimensional spacetime are described in terms of a Lorentzian metric tensor g and an independent metric compatible connection ω that is a rule for parallelly transporting tensorial quantities along curves in spacetime. The field equations satisfied by g and ω are obtained from a locally Lorentz invariant action

$$\int_{M_4} L(g, w)$$

by a well defined variational principle. Einstein gravitational theory is based on the single non-trivial curvature invariant linear in curvature components. Thus we consider the Einstein-Hilbert 4-action

$$I_0 = -\frac{1}{2\kappa^2}\int_{M_4} R_{ab} \wedge *(e^b \wedge e^b) \tag{1}$$

where $\kappa^2 = 8\pi G/c^3$ is the universal gravitational constant. Independent connection variations of I_0 imply that the connection is Levi-Civita. Then the source-free Einstein equations obtained by coframe variations of I_0 involve at most second-order partial derivatives of the components of the metric tensor.

Conceptually the simplest way to generalise Einstein theory is to write down 4-actions using higher order curvature invariants. A popular generalised theory of gravity is described by the 4-action [1, 2]

$$I, = k \int_{M_4} R_{ab} \wedge *R^{ab} \tag{2}$$

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that is written in analogy with Yang-Mills gauge theories. k is a dimensionless coupling constant. The field equations obtained by varying I_1, provided the spacetime torsion is constrained to be zero, involves at most fourth-order partial derivatives of the metric tensor components. It is no surprise that this theory admits non-physical solutions along with the physically admissible ones. Nevertheless, it is still an interesting theory. Firstly because it is motivated by an analogy with Yang-Mills theory, so that the gauge structure is manifest. Secondly, it leads to a renormalisable quantum gravity in a perturbative approach to field quantisation.

Electromagnetism is the only other long-range interaction in nature that is described by a classical field theory. The source-free Maxwell equations are obtained by a variational principle from the 4-action

$$I_{\mathrm{M}} = -\frac{1}{2e^2}\int_{M_4} F \wedge {}^*F \qquad (3)$$

where $F = \mathrm{d}A$ and e is the electric charge. The minimal coupling of electromagnetic fields to gravity defined by the 4-action $I_0 + I_{\mathrm{M}}$ provides a good description of the observed phenomena. Suppose we further add on higher derivative gravitational interactions by considering $I_0 + I_1 + I_{\mathrm{M}}$. In this case the minimal coupling rule fails. For instance, the external field of an isolated, static, spherically symmetric charge distribution can no longer be asymptotically flat. Can the electromagnetic fields be consistently coupled to generalised theories of gravity? This is our motivation for studying non-minimal electromagnetic couplings to higher derivative gravitational theories. In general at this level of generalisation we should have taken into account all possible curvature and U(1) gauge invariants that yield field equations involving at most fourth-order partial derivatives of the component fields. This would indeed be a very complicated theory. Is there any way to delineate some of these non-minimal couplings?

A different type of generalisation of Einstein's gravitational theory was pioneered by Kaluza and Klein who, being motivated by a desire to find a formal unification among the fundamental long-range forces of nature, considered the Einstein-Hilbert action over a five-dimensional spacetime manifold [3, 4]. They discovered that the Einstein-Hilbert 5-action can be reduced to the coupled Einstein-Maxwell system over the actual four-dimensional spacetime manifold. In this paper we apply the dimensional reduction technique of Kaluza-Klein to a higher derivative theory of gravitation and exhibit the non-minimal coupling of electromagnetic fields to gravity thus induced in four-dimensional actual spacetime.

2. Kaluza-Klein reduction

Let M_5 denote the five-dimensional spacetime manifold with topology $M_4 \times S^1$ where M_4 is the actual spacetime and S^1 is a compact internal space†. The radius of S^1 is usually assumed to be of the order of the Planck length. A global Killing vector denoted by K whose closed integral curves coincide with S^1 exists. Then the U_1 algebra generated by the action of K is related to the electromagnetic gauge invariance. The metric tensor on M_5 is given by

$$G = \eta_{AB} e^A \otimes e^B \qquad (4)$$

† Our notation and conventions are the same as those of [3].

where η_{AB} = diag$(-++++)$ and $\{e^A\}$ are the orthonormal coframes. The capital latin indices $A, B = 0, 1, 2, 3, 5$ refer to orthonormal frames. They are raised and lowered by η^{AB} and η_{AB}. We work in a coordinate chart $x^\mu : (x^\mu, y)$ which is adapted to the K isometry of (M_5, G) so that $K = \partial/\partial y$. Then the hypersurfaces y = constant are identified with M_4. The lower case latin indices $a, b = 0, 1, 2, 3$ refer to orthonormal frames on M_4. The SO(1, 4) Lie algebra valued connection l-forms $\{\Omega^A{}_B\}$ over M_5 are labelled so as to satisfy $\Omega_{AB} = -\Omega_{BA}$. The structure equations

$$\mathrm{d}e^A + \Omega^A{}_B \wedge e^B = T^A \qquad \mathrm{d}\Omega^A{}_B + \Omega^A{}_C \wedge \Omega^C{}_B = R^A{}_B \tag{5}$$

define the torsion 2-forms $T^A = T_{BC}{}^A e^B \wedge e^C$ and the curvature 2-forms $R^A{}_B = \frac{1}{2} R_{CD}{}^A{}_B e^C \wedge e^D$ on M_5. d denotes the exterior derivative, $\wedge$ the exterior product and $\#: E^p(M_5) \to E^{5-p}(M_5)$ is the Hodge map defined so that the invariant volume element $^\#1 = e^0 \wedge e^1 \wedge e^2 \wedge e^3 \wedge e^5$. The following choice of the orthonormal basis l-forms

$$e^a(x, y) = e^a(x) \qquad a = 0, 1, 2, 3 \qquad e^5(x, y) = \phi(x)(\mathrm{d}y + A(x)) \tag{6}$$

is consistent with the K isometry of the 5-metric. The substitution of (6) into *(4)* gives

$$G = g + \phi^2 A \otimes A + \phi(A \otimes \mathrm{d}y + \mathrm{d}y \otimes A) + \phi^2\, \mathrm{d}y \otimes \mathrm{d}y \tag{7}$$

from which we identify the 4-metric $g = \eta_{ab} e^a \otimes e^b$, the electromagnetic potential l-form $A = A_a e^a$, and a scalar field $\phi(x)$ on M_4. In fact ϕ is the variable norm of the Killing vector K. Given the coframe expression (6), the Levi-Civita connection l-forms are uniquely determined by solving the Cartan-Maurer equations (5) with torsion $T^A = 0$. We find

$$\Omega_{ab} = \omega_{ab} - \tfrac{1}{2}\phi F_{ab} e^5 \qquad \Omega_{5a} = -\Omega_{a5} = \tfrac{1}{2}\phi F_{ab} e^b + \frac{\partial_a \phi}{\phi} e^5. \tag{8}$$

Then the curvature 2-forms are given by

$$R_{ab} = \pi_{ab} + \tau_{ab} \wedge e^5 \qquad R_{5a} = -R_{a5} = \rho_a + \sigma_a \wedge e^5 \tag{9}$$

where

$$\begin{aligned}
\pi_{ab} &= R_{ab} - \tfrac{1}{2}\phi^2 F_{ab} F - \tfrac{1}{4}\phi^2 F_{ac} F_{bd} e^c \wedge e^d \\
\tau_{ab} &= -D(\tfrac{1}{2}\phi^2 F_{ab}) + \tfrac{1}{2} F_{ab}\, \mathrm{d}\phi + \tfrac{1}{2}(\partial_a \phi F_{bc} - \partial_b \phi F_{ac}) e^c \\
\rho_a &= D(\tfrac{1}{2}\phi F_{ab} e^b) + \partial_a \phi F \\
\sigma_a &= \frac{D(\partial_a \phi)}{\phi} + \tfrac{1}{4}\phi^2 F_{ac} F^c{}_b e^b.
\end{aligned} \tag{10}$$

We also need the following Hodge duality relations

$$^\#(e^a \wedge e^b) = *(e^a \wedge e^b) \wedge e^5 \qquad ^\#(e^a \wedge e^5) = *e^a \tag{11}$$

where $*: E^p(M_4) \to E^{4-p}(M_4)$ is the Hodge map defined with respect to the 4-metric g.

3. Non-minimal gauge couplings

We consider the higher derivative theory described by the 5-action [5, 6]

$$I = \int_{M_5} \left\{ k R_{AB} \wedge {}^\# R^{AB} - \frac{1}{2\kappa^2} R_{AB} \wedge {}^\#(e^A \wedge e^B) \right\}. \tag{12}$$

A few comments on the nature of variational field equations are in order. In general in any number of dimensions, the independent coframe and connection variations of a second order curvature invariant leads to a system of field equations which allows a dynamical spacetime torsion. In the present case we obtain the field equations

$$-\frac{1}{2\kappa^2}\boldsymbol{R}^{BC}\wedge{}^{\#}(\boldsymbol{e}_A\wedge\boldsymbol{e}_B\wedge\boldsymbol{e}_C)+\tfrac{1}{2}k(\iota_A\boldsymbol{R}_{BC}\wedge{}^{\#}\boldsymbol{R}^{BC}-\boldsymbol{R}_{BC}\wedge\iota_A{}^{\#}\boldsymbol{R}^{BC})=0 \tag{13}$$

by coframe variations and

$$-\frac{1}{2\kappa^2}{}^{\#}(\boldsymbol{e}_A\wedge\boldsymbol{e}_B\wedge\boldsymbol{e}_C)\wedge\boldsymbol{T}^C+2k\boldsymbol{D}^{\#}\boldsymbol{R}_{AB}=0 \tag{14}$$

by connection variations. It is apparent from those equations that the spacetime torsion need not necessarily vanish. Then it is possible for dimensional reduction to prescribe independent torsion degrees of freedom, provided they satisfy K-isometry conditions

$$\mathscr{L}_K(\boldsymbol{T}^A\otimes\boldsymbol{X}_A)=0 \tag{15}$$

where $\mathscr{L}_K$ denotes a Lie derivative with respect to $\boldsymbol{K}$ and $\{\boldsymbol{X}_A\}$ are the frame fields such that $\boldsymbol{G}(\boldsymbol{X}_A,\boldsymbol{X}_B)=\eta_{AB}$. Here we consider only the case of vanishing spacetime torsion, and we constrain our variations to preserve this choice. This can be achieved, for instance, by the method of Lagrange multipliers and the field equations we obtain from (12) are given as

$$-4k\boldsymbol{D}(\iota^B\boldsymbol{D}^{\#}\boldsymbol{R}_{AB})+2k\boldsymbol{e}_A\wedge\boldsymbol{D}(\iota^C\iota^B(\boldsymbol{D}^{\#}\boldsymbol{R}_{BC}))+\tfrac{1}{2}k(\iota_A\boldsymbol{R}_{BC}\wedge{}^{\#}\boldsymbol{R}^{BC}-\boldsymbol{R}_{BC}\,A\,\iota_A{}^{\#}\boldsymbol{R}^{BC})$$

$$-\frac{1}{2\kappa^2}\boldsymbol{R}^{BC}\wedge{}^{\#}(\boldsymbol{e}_A\wedge\boldsymbol{e}_B\wedge\boldsymbol{e}_C)=0 \tag{16}$$

subject to the constraint that the connection is Levi-Civita.

Our remaining task is to substitute the curvature 2-forms (9) together with Hodge duality relations (11) into the **5-action** (12) and obtain a reduced **4-action** density defined by

$$L_5=\mathrm{d}y\wedge L_4(g,\phi,A). \tag{17}$$

We performed the algebraic manipulations necessary for dimensional reduction on computer using the exterior calculus package XTR for REDUCE [7]. We obtained the following expression:

$$\boldsymbol{L_4}=\phi R_{ab}\,A\,{}^*R^{ab}+\tfrac{1}{2}\alpha\phi R_{ab}\,A\,{}^*e^{ab}-\tfrac{1}{4}\alpha\phi^3F\wedge{}^*F-\alpha d{}^*d\phi-\tfrac{3}{2}\phi^3R_{ab}\,A\,F^{ab}{}^*F$$

$$+\tfrac{3}{16}\phi^5(F_{ab})^{4*}1+\tfrac{5}{16}\phi^5F_{ab}F^{bc}F_{cd}F^{da*}1+\tfrac{1}{4}\phi D(\phi F_{ab})\,\mathrm{A}\,{}^*D(\phi F^{ab})$$

$$+\tfrac{1}{2}\phi D(\phi F_{ab}e^b)\wedge{}^*(D(\phi F^a{}_ce^c))$$

$$+\frac{1}{2\phi}D(\partial_a\phi)\wedge{}^*D(\partial^a\phi)+\tfrac{7}{2}F\wedge{}^*F(\partial_a\phi)^2\phi$$

$$-\tfrac{1}{2}\phi F_{ab}F^b{}_c\partial^a\phi\partial^c\phi+\tfrac{1}{2}\phi^2F^{ab}\,\mathrm{d}\phi\,\mathrm{A}\,{}^*DF_{ab}+\phi F_{ac}e^c\,\mathrm{A}\,{}^*D(\phi F^{ab})\partial_b\phi$$

$$\text{-I-2°}F\ \mathrm{A}\,D(\phi F_{ab}e^b)\partial^a\phi\phi^2+\phi^2F_{ab}F^b{}_ce^c\,\mathrm{A}\,{}^*D(\partial^a\phi) \tag{18}$$

where we set the coupling constant $\alpha=-1/2k\kappa^2$. Note that the kinetic terms for the photon and the scalar boson fields implicitly contain couplings to curvature components. In order to make these couplings manifest, we must write these terms in a way independent of the choice of frame fields. The simplifications leading to this goal involve, first, the use of gravitational and gauge Bianchi identities, and second, partial differentiations resulting in closed forms that do not affect the variational field equations. We proved the identities (modulo closed forms)

$$\begin{aligned} D(\phi F^{ab}) \wedge {}^*D(\phi F^{ab}) &= -2\mathrm{d}^*(\phi F) \wedge {}^*(\mathrm{d}^*(\phi F)) + 2\phi^2 R_{ab} \wedge F^{ab*} F \\ &\quad - 2F_{ac}e^c \wedge {}^*D(\phi F^{ab})\partial_b\phi + F_{ab}\,\mathrm{d}\phi \wedge {}^*D(\phi F^{ab}) \\ &\quad - 4\phi^2 P^a \wedge (\iota_a F \wedge {}^*F - F \wedge \iota_a{}^*F) - \phi^2 Q F \wedge {}^*F \end{aligned} \tag{19}$$

and

$$\begin{aligned} D(\phi F^e{}_{ab} b) \wedge {}^* D(\phi F^a{}_c e^c) &= \tfrac{1}{2} D(\phi F^{ab}) \wedge {}^*D(\phi F^{ab}) - F_{ac}e^c \wedge {}^*D(\phi F^{ab})\partial_b\phi \\ &\quad + \tfrac{1}{2}F_{ab}\,\mathrm{d}\phi \wedge {}^*D(\phi F^{ab}). \end{aligned} \tag{20}$$

Here $P_a = \mathcal{R}_{ba}e^b$ are the Ricci 1-forms and $Q = \eta^{ab}\mathcal{R}_{ab}$ is the curvature scalar. We also made use of the identity

$$\begin{aligned} D(\partial_a\phi) \wedge {}^*D(\partial^a\phi) &= -\mathrm{d}\,{}^*\mathrm{d}\phi \wedge {}^*(\mathrm{d}\,{}^*\mathrm{d}\phi) \\ &\quad - \tfrac{1}{2}P^a \wedge (\partial_a\phi\,{}^*\mathrm{d}\phi + \mathrm{d}\phi \wedge \iota_a{}^*\mathrm{d}\phi) - \tfrac{1}{2}Q\,\mathrm{d}\phi \wedge {}^*\mathrm{d}\phi. \end{aligned} \tag{21}$$

Substituting the above identities in (18) and reorganising terms we reach the following expression for the reduced action density:

$$\begin{aligned} L_4 = {} & \phi R_{ab} \wedge {}^*R^{ab} + \tfrac{1}{2}\alpha\phi R_{ab} \wedge {}^*e^{ab} - \phi\,\mathrm{d}^*(\phi F) \wedge {}^*(\mathrm{d}^*(\phi F)) - \tfrac{1}{4}\alpha\phi^3 F \wedge {}^*F \\ & - \tfrac{11}{8}\phi^5 (F \wedge {}^*F)^*(F \wedge {}^*F) + \tfrac{5}{16}\phi^5 (F \wedge F)^*(F \wedge F) \\ & - \tfrac{1}{2}\phi^3 R_{ab} \wedge F^{ab*}F - 2\phi^3 P^a \wedge (\iota_a F \wedge {}^*F - F \wedge \iota_a{}^*F) - \tfrac{1}{2}\phi^3 Q F \wedge {}^*F \\ & - \frac{2}{\phi}\,\mathrm{d}\,{}^*\mathrm{d}\phi\,{}^*(\mathrm{d}\,{}^*\mathrm{d}\phi) - \alpha\,\mathrm{d}\,{}^*\mathrm{d}\phi \\ & - P^a \wedge \frac{(\partial_a\phi\,{}^*\mathrm{d}\phi + \mathrm{d}\phi \wedge \iota_a{}^*\mathrm{d}\phi)}{\phi} - Q\frac{\mathrm{d}\phi \wedge {}^*\mathrm{d}\phi}{\phi} \\ & + 4(F_{ab})^2\phi\,\mathrm{d}\phi \wedge {}^*\mathrm{d}\phi + 4F^a{}_b F_{ac}\partial^b\phi\partial^c\phi\phi{}^*1 + 3\phi^2\,\mathrm{d}\phi \wedge F_{ab}{}^*DF^{ab}. \end{aligned} \tag{22}$$

We can now read-off from (22) various types of interactions between the (spin-2) graviton field g, (spin-1) photon field F, and (spin-0) scalar boson field ϕ. The obvious ground state of the above system is fixed by setting $g = \eta$, $F = 0$ and $\phi = 1$. Then different types of interactions are distinguished as follows:
For the case $F = 0$, $\phi = 1$ we get pure gravitational interactions [8, 9, 10]:

$$L_g = R_{ab} \wedge {}^*R^{ab} + \tfrac{1}{2}\alpha R_{ab} \wedge {}^*e^{ab}. \tag{23}$$

For the case $g = \eta, \phi = 1$ we get pure electromagnetic interactions:

$$L_F = -\mathrm{d}^*F \wedge {}^*(\mathrm{d}^*F) - \tfrac{1}{4}\alpha F \wedge {}^*F - \tfrac{11}{8}(F \wedge {}^*F)^*(F \wedge {}^*F) + \tfrac{5}{16}(F \wedge F)^*(F \wedge F). \tag{24}$$

The first two terms correspond to a generalised theory of electromagnetism considered by Bopp [11] and Podolsky [12]. The last two terms are a particular combination of quartic Maxwell invariants. For the case $g = \eta, F = 0$ we get the kinetic term for the scalar boson:

$$L_\phi = -\frac{2}{\phi}\, \mathrm{d}\, {*\mathrm{d}\phi}\, {*(\mathrm{d}\, {*\mathrm{d}\phi})} - \alpha\, \mathrm{d}\, {*\mathrm{d}\phi}. \tag{25}$$

The graviton-photon interaction terms are found by setting $\phi = 1$:

$$L_{g-F} = -\tfrac{1}{2}R_{ab} \wedge F^{ab} {*F} - 2P^a \wedge (\iota_a F \wedge {*F} - F \wedge \iota_a {*F}) - \tfrac{1}{2}QF \wedge {*F}. \tag{26}$$

The first term corresponds to the direct curvature-electromagnetic-field coupling studied by Prassana [13] and Buchdahl [14]. The other terms involve the coupling of the Ricci tensor to the electromagnetic stress-energy-momentum tensor and are new. The graviton-scalar boson interactions, found by setting $F = 0$, have a similar form:

$$L_{g-\phi} = -P^a \wedge \frac{(\partial_a \phi\, {*\mathrm{d}\phi} + \mathrm{d}\phi \wedge \iota_a {*\mathrm{d}\phi})}{\phi} - Q\frac{\mathrm{d}\phi \wedge {*\mathrm{d}\phi}}{\phi}. \tag{27}$$

Finally the photon-scalar boson interactions are determined by setting $g = \eta$:

$$L_{\phi-F} = 4(F_{ab})^2 \phi\, \mathrm{d}\phi \wedge {*\mathrm{d}\phi} + 4F^a{}_b F_{ac} \partial^b \phi \partial^c \phi\, \phi\, {*1} + 3\phi^2\, \mathrm{d}\phi \wedge F_{ab} {*DF^{ab}}. \tag{28}$$

4. Concluding remarks

The main result of this paper is the reduced action density given by equation (22). Some of the interactions implied by this expression are new, and might deserve a further scrutiny.

We would like to emphasise once again that the introduction of independent torsion degrees of freedom is allowed. For instance, we may start by choosing, together with the Kaluza-Klein metric ansatz (7), $T^5 = \phi F$ and $T^a = 0$. There will be many other choices, implied by condition (15) but we won't be studying this question here any further.

Kaluza-Klein reduction of other generalised theories of gravity might be considered. In fact the dimensionally reduced form of the Euler-Poincare S-action has recently been given [15, 16].

The algebraic manipulations leading to the Kaluza-Klein reduced 4-action (18) are performed on computer using the exterior calculus package XTR for REDUCE. Our dimensional reduction algorithm is prepared to handle more general cases. It will be able to dimensionally reduce gravitational actions in an arbitrarily large but fixed number of dimensions with more complicated metric structure than we used here.

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References

[1] Benn I M, Dereli T and Tucker R W 1982 *J. Phys. A: Math. Gen.* 15 849
[2] Mielke E 1987 *Geometrodynamics of Gauge Fields* (Berlin: Akademie) and references therein
[3] Dereli T and Tucker R W 1982 *Nucl. Phys.* B *209 217*
[4] Appelquist T, Chodos A and Freund P G 0 1987 *Modem Kaluza-Klein Theories* (Reading, MA: Addison-Wesley) and references therein

[5] Dereli T and Sualp G 1983 ***Proc. 10th Int.Conf. on General Relativity and Gravitation*** ed B Bertotti, T De Felice and A Pascolini (Padova) p 501
[6] Ross D K 1983 ***J. Phys. A: Math. Gen. 16 3879***
[7] Dereli T and Üçoluk G ***J. Comput. Phys.*** to be published
[8] Stephenson G 1958 Nuovo ***Cimento 2 263***
[9] Kilmister C W and Newman D J 1961 ***Proc. Camb. Phil. Soc. 57*** 851
[10] Benn I M, Dereli T and Tucker R W 1981 ***Gen. Rel. Grav. 13*** 581
[11] Bopp F 1940 ***Ann. Phys., Lpz. 38 345***
[12] Podolsky B 1941 ***Phys. Rev. 62 68***
[13] Prassana A R 1971 ***Phys. Lett. 37A 331***
[14] Buchdahl H A 1979 ***J. Phys. A: Math. Gen. 12 1037***
[15] Kerner R 1986 ***Proc. 7th Italian Conf. on General Relativity and Gravitational Physics*** ed U Bruzzo, R Cianci, E Massa (Singapore: World Scientific) p 213
[16] Muller-Hoissen F 1988 ***Phys. Lett. 201B 325***