Direct-curvature Yang-Mills field couplings induced by the Kaluza-Klein reduction of Euler form actions in seven dimensions

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Abstract. The Kaluza-Klein reduction of Euler form actions on $M_4 \times S^3$ is considered. Direct-curvature Yang-Mills field coupling terms in the reduced action of the generic types RF^2 and RF^3 are explicitly given.

1. Introduction

A formal unification of long-range interactions of nature can be achieved by constructing gravitational field theories in higher dimensional spacetimes [1]. The types of force fields and the form of interactions among them are determined by the dimensionally reduced theory in physical four-dimensional spacetime. In four dimensions, experimental evidence shows that the Einstein field equations

$$G_{ab} = -\kappa^2 T_{ab} \tag{1}$$

adequately describe the observed phenomena. Here T_{ab} are the components of the symmetrised stress-energy-momentum tensor of the matter fields, $\kappa^2 = 8\pi G/c^3$ is the universal gravitational coupling constant and G_{ab} denotes the covariant components of the second rank symmetric Einstein tensor. The latter has the unique property of being covariantly constant and involving at most second-order partial derivatives of the metric tensor components. Further, we know that the Einstein tensor follows from the local variations of the Einstein-Hilbert action which is linear in curvature components:

$$I_E = -\frac{1}{2\kappa^2}\int_{M_4} R_{ab} \wedge *(e^b \wedge e^b). \tag{2}$$

In spacetime dimensions $D > 4$, there are other higher-order curvature invariants which, when used in an action, contribute to gravitational field equations, partial derivatives of metric components of order no higher than two [2]. In fact, these higher-order curvature invariants can be given in terms of dimensionally continued Euler forms [3]

$$L_D^{(n)} = R_{a_1b_1} \wedge R_{a_2b_2} \wedge \ldots \wedge R_{a_nb_n} *(e^{a_1} \wedge e^{b_1} \wedge \ldots \wedge e^{a_n} \wedge e^{b_n}) \tag{3}$$

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in dimensions $D > 2n, n = 0, 1, \ldots, [D/2]$. For the case $D = 2n, L^{(n)}_{2n}$ is an exact form. Its integral over the spacetime manifold M_{2n} is proportional to the topological Euler-Poincare characteristic $\chi(M_{2n})$.

Dimensional reduction of the second-order Euler-Poincare action and some of its physical consequences are already discussed in the literature [4]. In general, it was shown [5] that the Kaluza-Klein reduction of $L^{(n)}_D$ in $D = 2n + 3$ dimensions $(n > 1)$ down to $D = 4$ dimensions gives a vanishing cosmological term besides the Einstein-Yang-Mills term, disregarding all other higher-order contributions to the reduced action. It was suggested earlier that in $D > 2n$ dimensions, a gravitational action consisting of a linear combination of all non-trivial Euler forms should be used [6]:

$$I = \sum_{n=0}^{[D/2]} k_n \int_{M_4} Lb"'. \tag{4}$$

In fact this idea was motivated by a study of string-induced gravitational models [7]. However, it should be noted that in those models the actual gauge fields can be related with degrees of freedom other than the spacetime metric. In a previous study [8] we showed by explicit calculation that gravitational and gauge field interactions induced by the Kaluza-Klein reduction of higher-dimensional gravitational theories based on quadratic curvature invariants can be quite complicated. There exist in the literature explicit calculations displaying several types of gravitational and gauge field couplings induced by the dimensional reduction of Euler form actions [9]. However, a complete Kaluza-Klein reduction of *(4)* has not yet been given. In this paper we consider the case $D = 7$ and dimensionally reduce the 7-action (4) on $M_4 \times S^3$. As S^3 is the group manifold of SO(3), the off-diagonal components of the Kaluza-Klein 7-metric are identified with the Yang-Mills potentials. The actual gravitational and Yang-Mills field interactions are then determined from the reduced 4-action. Straightforward but tedious generalisations, such as those involving an arbitrary gauge group G, or those including a scalar field that describes a variable radius for S^3, will not be attempted here.

2. **Notation and conventions**

The gravitational fields in seven-dimensional spacetime are described by the Lorentzian metric

$$G = \eta_{AB} e^A \otimes e^B \tag{5}$$

where η_{AB} = diag($-$ + + + + + +) and a set of independent but metric compatible connection l-forms $\{\Omega^A{}_B\}$. The indices $A, B, \ldots = 0, 1, 2, 3, 5, 6, 7$ refer to orthonormal frame $\{X_A\}$. They are raised and lowered by η^{AB} and η_{AB}. $\{e^A\}$ are the orthonormal basis l-forms dual to the frame fields, i.e. $e^A(X_B) = \delta^A{}_B$. They satisfy the structure equations

$$\mathrm{d}e^A + \Omega^A{}_B \wedge e^B = T^A \tag{6}$$

$$\mathrm{d}\Omega^A{}_B + \Omega^A{}_C \,\mathrm{A}\, \Omega^C{}_B = R^A{}_B \tag{7}$$

where $T^A = T_{BC,}{}^A e^B \,\mathrm{A}\, e^C$ are the torsion 2-forms and $R^A{}_B = \frac{1}{2} R_{CD,}{}^A{}_B\, e^C \,\mathrm{A}\, e^D$ are the curvature 2-forms of the spacetime. The integrability conditions for the structure equations yield the Bianchi identities

$$\mathrm{d}T^A + \Omega^A{}_B \wedge T^B = R^A{}_B \wedge e^B \tag{8}$$

$$\mathrm{d}R^A{}_B + \Omega^A{}_C \,\mathrm{A}\, R^C{}_B + \Omega_B{}^C \,\mathrm{A}\, R^A{}_C = 0. \tag{9}$$

The following linear operators on forms are defined. $d \equiv e^a \nabla_X : E^p(M) \to E^{p+1}(M)$ *is* the exterior derivative. $\iota_A : E^p(M) \to E^{p-1}(M)$ are the interior product operators such that $\iota_A(e^B) = \delta_A{}^B$. We define also the Hodge map $\# : E^p(M) \to E^{7-p}(M)$ defined so that the invariant volume element

$$\# 1 = e^0 \wedge e^1 \wedge e^2 \wedge e^3 \wedge e^5 \wedge e^6 \wedge e^7.$$

3. **Kaluza-Klein reduction on $M_4 \times S^3$**

We will work in a coordinate chart $x^M : (x^\mu, y^m)$ adopted to the isometries of spacetime. Here $(x^\mu), \mu = 0, 1, 2, 3$, are the coordinates of the 4-spacetime, and (y^m), $m = 5, 6, 7$, constitute a coordinate system for S^3. Suppose $\{e^\alpha\}$ is a set of orthonormal basis 1-forms on S^3. Then the metric on the 3-sphere

$$g_{S^3} = \delta_{\alpha\beta} e^\alpha \otimes e^\beta = g_{mn}(y)\, dy^m \otimes dy^n \tag{10}$$

and the dual frame vectors $\{X_\alpha\}$ such that $e^\alpha(X_\beta) = \delta^\alpha{}_\beta$ generate the Lie algebra of SO(3)

$$[X_\alpha, X_\beta] = \varepsilon_{\alpha\beta}{}^\gamma X_\gamma. \tag{11}$$

We have the structure equations

$$de^\alpha = \tfrac{1}{2}\varepsilon^\alpha{}_{\beta\gamma} e^\beta \wedge e^\gamma. \tag{12}$$

The Kaluza-Klein ansatz for the basis 1-forms read

$$\begin{aligned} e^a &= e^a(x) \qquad a = 0, 1, 2, 3 \\ e^\alpha &= e^\alpha(y) + A^\alpha(x) \qquad \alpha = 5, 6, 7. \end{aligned} \tag{13}$$

The actual spacetime 4-metric

$$g = \eta_{ab} e^a \otimes e^b \tag{14}$$

and the Levi-Civita 4-connections are found by solving the structure equations

$$de^a + \omega^a{}_b \wedge e^b = 0. \tag{15}$$

We identify $A^\alpha = A^\alpha{}_a e^a$ with the Yang-Mills potential 1-forms so that the Yang-Mills field 2-forms

$$F^\alpha = dA^\alpha + \tfrac{1}{2}\varepsilon^\alpha{}_{\beta\gamma} A^\beta \wedge A^\gamma \equiv \tfrac{1}{2} F^\alpha_{ab} e^a \wedge e^b. \tag{16}$$

They satisfy the gauge Bianchi identity

$$D_A F^\alpha \equiv dF^\alpha + \varepsilon^\alpha{}_{\beta\gamma} A^\beta \wedge F^\gamma = 0. \tag{17}$$

We substitute the Kaluza-Klein ansatz (13) in (6) (with vanishing torsion) and solve for the connection 1-forms:

$$\Omega_{ab} = \omega_{ab} - \tfrac{1}{2} F^\alpha_{ab} e_\alpha \tag{18}$$

$$\Omega_{\alpha\beta} = \tfrac{1}{2}\varepsilon_{\alpha\beta\gamma}(e^\gamma - A^\gamma) \tag{19}$$

$$\Omega^\alpha{}_a = -\Omega_a{}^\alpha = \tfrac{1}{2} F^\alpha_{ab} e^b. \tag{20}$$

Then putting the connection 1-forms given above into (7), we find the curvature 2-forms

$$\boldsymbol{R}_{ab} = \pi_{ab} + \tau^{\alpha}_{ab} \wedge \boldsymbol{e}_{\alpha} + \Sigma_{ab}{}^{\alpha\beta}\boldsymbol{e}_{\alpha} \wedge \boldsymbol{e}_{\beta} \tag{21}$$

$$\boldsymbol{R}_{\alpha\beta} = \Psi_{\alpha\beta} + \tfrac{1}{4}\boldsymbol{e}_{\alpha} \wedge \boldsymbol{e}_{\beta} \tag{22}$$

$$\boldsymbol{R}^{\alpha}{}_{a} = -\boldsymbol{R}_{a}{}^{\alpha} = \rho_{a}{}^{\alpha} + \sigma_{a}{}^{\alpha}{}_{\beta} \wedge \boldsymbol{e}^{\beta} \tag{23}$$

where

$$\pi_{ab} = R_{ab} - \tfrac{1}{2}F^{\alpha}_{ab}F^{\alpha} - \tfrac{1}{4}F^{\alpha}_{ac}e^{c} \wedge F_{\alpha bd}e^{d} \tag{24}$$

$$\tau^{\alpha}_{ab} = -\tfrac{1}{2}D_{A}F^{\alpha}_{ab} \tag{25}$$

$$\Sigma_{ab}{}^{\alpha\beta} = -\tfrac{1}{4}\varepsilon^{\alpha\beta}{}_{\gamma}F^{\gamma}_{ab} + \tfrac{1}{4}F^{\alpha}_{ac}F^{\beta c}{}_{b} \tag{26}$$

$$\Psi^{ab} = -\tfrac{1}{2}\varepsilon^{\alpha\beta}{}_{\gamma}F^{\gamma} + \tfrac{1}{4}F^{\alpha}_{ac}F^{\beta c}{}_{b}e^{a} \wedge e^{b} \tag{27}$$

$$\rho_{a}{}^{\alpha} = \tfrac{1}{2}(D_{A}F^{\alpha}_{ab}) \wedge e^{b} \tag{28}$$

$$\sigma_{a}{}^{\alpha}{}_{\beta} = \tfrac{1}{4}\varepsilon^{\alpha}{}_{\beta\gamma}F^{\gamma}_{ab}e^{b} + \tfrac{1}{4}F^{\alpha}_{cb}F_{\beta}{}^{b}{}_{a}e^{c}. \tag{29}$$

For the purpose of dimensional reduction, we further need the following decomposition of the Hodge map:

$$^{\#}1 = {}^{*}1 \wedge e^{5} \wedge e^{6} \wedge e^{7} \tag{30}$$

where $^{*}: E^{p}(M) \to E^{4-p}(M)$ denotes the Hodge map with respect to the 4-metric g. Then a dimensionally reduced 4-action density will be defined from the identity

$$L_{7} = L_{4} \wedge e^{5} \wedge e^{6} \wedge e^{7}. \tag{31}$$

We substitute the curvature 2-forms (21)–(23) in the dimensionally continued Euler densities (3) for the cases n = 1, 2, 3 and find the following reduced 4-action densities. We checked our calculations on computer using the exterior calculus package XTR in **REDUCE** [10].

$$L_{4}^{(1)} = \pi_{ab} \wedge {}^{*}(e^{a} \wedge e^{b}) - 2\sigma_{a}{}^{\alpha}{}_{\alpha} \wedge {}^{*}e^{a} + \tfrac{3}{3}{}^{*}1 \tag{32}$$

$$\begin{aligned} L_{4}^{(2)} = {} & \varepsilon^{abcd}\pi_{ab} \wedge \pi_{cd} - 4(\rho_{a}{}^{\alpha} \wedge \tau_{bc\alpha} + \sigma_{a}{}^{\alpha}{}_{\alpha} \wedge \pi_{bc}) \wedge {}^{*}(e^{a} \wedge e^{b} \wedge e^{c}) + 3\pi_{ab} \wedge {}^{*}(e^{a} \wedge e^{b}) \\ & + 4\Sigma_{ab}{}^{\alpha\beta}\Psi_{\alpha\beta} \wedge {}^{*}(e^{a} \wedge e^{b}) - 4\sigma_{a}{}^{\alpha}{}_{\alpha} \wedge \sigma_{b}{}^{\beta}{}_{\beta} \wedge {}^{*}(e^{a} \wedge e^{b}) \\ & + 4\sigma_{a}{}^{\alpha}{}_{\beta} \wedge \sigma_{b}{}^{\beta}{}_{\alpha} \wedge {}^{*}(e^{a} \wedge e^{b}) - 2\sigma_{a}{}^{\alpha}{}_{\alpha} \wedge {}^{*}e^{a} \end{aligned} \tag{33}$$

$$\begin{aligned} L_{4}^{(3)} = {} & \tfrac{9}{2}\varepsilon^{abcd}\pi_{ab} \wedge \pi_{cd} + 12\varepsilon^{abcd}\pi_{ab} \wedge \Sigma_{cd}{}^{\alpha\beta}\Psi_{\alpha\beta} + 6\varepsilon^{abcd}\tau^{\alpha}_{ab} \wedge \tau^{\beta}_{cd} \wedge \Psi_{\alpha\beta} \\ & - 24\varepsilon^{abcd}\Sigma_{ab\alpha\beta}\rho_{c}{}^{\alpha} \wedge \rho_{d}{}^{\beta} - 24\varepsilon^{abcd}\tau_{ab\alpha} \wedge \rho_{c}{}^{\alpha} \wedge \sigma_{d}{}^{\beta}{}_{\beta} + 24\varepsilon^{abcd}\tau_{ab\alpha} \wedge \rho_{c}{}^{\beta} \wedge \sigma_{d}{}^{\alpha}{}_{\beta} \\ & - 12\varepsilon^{abcd}\pi_{ab} \wedge \sigma_{c}{}^{\alpha}{}_{\alpha} \wedge \sigma_{d}{}^{\beta}{}_{\beta} + 12\varepsilon^{abcd}\pi_{ab} \wedge \sigma_{c}{}^{\alpha}{}_{\beta} \wedge \sigma_{d}{}^{\beta}{}_{\alpha} \\ & + 6\pi_{ab} \wedge {}^{*}(e^{a} \wedge e^{b} \wedge e^{c}) \wedge \sigma_{c}{}^{\alpha}{}_{\alpha} + 6\tau_{ab\alpha} \wedge {}^{*}(e^{a} \wedge e^{b} \wedge e^{c}) \wedge \rho_{c}{}^{\alpha} \\ & + 12\varepsilon_{\alpha\beta\gamma}\varepsilon^{\alpha'\beta'\gamma'}\Sigma_{ab\alpha'\beta'}{}^{*}(e^{a} \wedge e^{b} \wedge e^{c}) \wedge \sigma_{c}{}^{\alpha}{}_{\gamma'} \wedge \Psi^{\beta\gamma} \\ & + 8\varepsilon_{\alpha\beta\gamma}\varepsilon^{\alpha'\beta'\gamma'}\sigma_{a}{}^{\alpha}{}_{\alpha'} \wedge \sigma_{b}{}^{\beta}{}_{\beta'} \wedge \sigma_{c}{}^{\gamma}{}_{\gamma'}{}^{*}(e^{a} \wedge e^{b} \wedge e^{c}). \end{aligned} \tag{34}$$

Our final task is to substitute the expressions (24)–(29) in the above formulae. The dimensionally reduced Einstein-Hilbert action density needs no further discussion [11]:

$$L_{4}^{(1)} = R_{ab} \wedge {}^{*}(e^{a} \wedge e^{b}) - \tfrac{1}{2}F_{\alpha} \wedge {}^{*}F^{\alpha} + \tfrac{3}{2}{}^{*}1. \tag{35}$$

Here the cosmological constant is proportional to the SO(3) group space volume. Introducing back physical units, the value of the cosmological constant induced by dimensional reduction turns out to be unacceptably large. The dimensional reduction of the second order Euler-Poincare density gives (modulo a closed form) [9]:

$$L_4^{(2)} = 3R_{ab} \wedge *(e^a \wedge e^b) - \tfrac{5}{2}F_\alpha \wedge *F^\alpha - \tfrac{3}{2}\varepsilon_{\alpha\beta\gamma}F^\alpha{}_a{}^b F^\beta{}_b{}^c F^\gamma{}_c{}^a *1$$
$$- 5R_{ab}F_\alpha{}^{ab} \wedge *F^\alpha + 5P^a \wedge (\iota_a F^\alpha \wedge *F_\alpha - F^\alpha \wedge \iota_a *F_\alpha)$$
$$- 2(F^\alpha \wedge *F_\alpha)*(F^\beta \wedge *F_\beta) + 2(F_\alpha \wedge *F_\beta)*(F^\alpha \wedge *F^\beta)$$
$$+ \tfrac{1}{2}(F_\alpha \wedge F_\beta)*(F^\alpha \wedge F^\beta) - \tfrac{1}{8}(F^\alpha \wedge F_\alpha)*(F_\beta \wedge F^\beta). \tag{36}$$

Just as we expected the above expression does not contain any higher-derivative couplings. At the lowest order of approximation, except for unusual scale factors, it consists of the Einstein-Yang-Mills 4-action with vanishing cosmological constant. In the next order of approximation we find an F^3 term [12]. The presence of direct-curvature couplings to Yang-Mills fields of generic form RF^2 has already been pointed out. The other direct-curvature Yang-Mills field couplings induced by the Kaluza-Klein reduction of $L_7^{(3)}$ are new, and here we wish to concentrate on those. Going back to expression (34) and checking the form of the functions (24)-(29), we observe that the generic types of interactions we would get in the reduced 4-action density will be $RF^2, F^3, F^4, RF^3, RF^4, F^5$ and F^6. In fact we find terms of the type $(DF)^2F$, but these yield upon partial differentiation and using gauge and gravitational Bianchi identities, terms of the generic types RF^3 and F^4. In a similar way, terms of the type $(DF)^2F^2$ will be replaced by terms of the type RF^4 and F^5. The RF^2 couplings thus induced from the reduction of $L_7^{(3)}$ are found to be

$$\tfrac{21}{2}R_{ab}F_\alpha{}^{ab} \wedge *F^\alpha + 12P^a \wedge (\iota_a F^\alpha \wedge *F_\alpha - F^\alpha \wedge \iota_a *F_\alpha) + \tfrac{3}{2}\mathcal{Q}F^\alpha \wedge *F_\alpha. \tag{37}$$

Similarly the F^3 term found by reducing $L_7^{(3)}$ is

$$3\,\varepsilon_{\alpha\beta\gamma}F^\alpha{}_a{}^b F^\beta{}_b{}^c F^\gamma{}_c{}^a *1. \tag{38}$$

These together with F^4-type coupling obtained from $L_7^{(3)}$, are added onto expression (36), thus giving rise to shifts in the corresponding coupling coefficients.

4. RF^3-type couplings

The simplest non-trivial direct-curvature Yang-Mills field couplings induced by the Kaluza-Klein reduction of the third-order Euler-Poincare density are of the generic type RF^3. The derivation of the actual form of these couplings is rather involved, so we give some details. Explicit RF^3 couplings come from the following terms in (34):

$$12\varepsilon^{abcd}\pi_{ab} \wedge \Sigma_{cd}{}^{\alpha\beta}\Psi_{\alpha\beta} - 12\varepsilon^{abcd}\pi_{ab} \wedge \sigma_c{}^\alpha{}_\alpha \wedge \sigma_d{}^\beta{}_\beta + 12\varepsilon^{abcd}\pi_{ab} \wedge \sigma_c{}^\alpha{}_\beta \wedge \sigma_d{}^\beta{}_\alpha.$$

Substituting from (24)-(29) and keeping only the RF^3-type terms we see that the reduced 4-action density will get the contribution

$$-6R^*_{ab} \wedge F^{\alpha a}{}_c e^c \wedge H_\alpha{}^b{}_d e^d - 3R_{ab} \wedge F^{\alpha ab}H_\alpha - 6R_{ab} \wedge H^{\alpha ab}F_\alpha \tag{39}$$

where $R^*_{ab} = \tfrac{1}{2}\varepsilon_{ab}{}^{cd}R_{cd}$ and we defined the 2-forms

$$H^\alpha = \tfrac{1}{2}\varepsilon^\alpha{}_{\beta\gamma}F^\beta_{ac}F^{\gamma c}{}_b e^a \wedge e^b \equiv \tfrac{1}{2}H^\alpha_{ab}e^a \wedge e^b. \tag{40}$$

Next we make use of the gravitational Bianchi identity $R^a{}_b \wedge e^b = 0$ and cast expression (39) into the form

$$-24R^*_{ab} \wedge H^{\alpha ab}F_\alpha. \tag{41}$$

This may as well be written in an equivalent way

$$6R^{ab} \wedge H^\alpha_{ab}{}^*F_\alpha + 24\mathcal{R}^{ab}F^\alpha_{ac}H_\alpha{}^c{}_b{}^*1 + 12\mathcal{Q}H^\alpha \wedge {}^*F_\alpha \tag{42}$$

where $\mathcal{R}_{ab}e^b = \iota^b R_{ba}$ are the Ricci l-form and $\mathcal{Q} = \iota^a\iota^b R_{ba}$ is the curvature scalar. The remaining RF^3 couplings are implicit in $(DF)^2F$ type interactions which come from the following terms in (34):

$$6\varepsilon^{abcd}\tau^\alpha_{ab} \wedge \tau^\beta_{cd} \wedge \Psi_{\alpha\beta} - 24\varepsilon^{abcd}\Sigma_{ab\alpha\beta}\rho_c{}^\alpha \wedge \rho_d{}^\beta$$
$$- 24\varepsilon^{abcd}\tau_{ab\alpha} \wedge \rho_c{}^\alpha \wedge \sigma_d{}^\beta{}_\beta + 24\varepsilon^{abcd}\tau_{ab\alpha} \wedge \rho_c{}^\beta \wedge \sigma_d{}^\alpha{}_\beta.$$

Substituting above from the expressions (24)–(29) we find the relevant terms to be given by

$$\tfrac{3}{4}\varepsilon^{abcd}D_AF^\alpha_{ab} \wedge D_AF^\beta_{cd} \wedge \varepsilon_{\alpha\beta\gamma}F^\gamma - \tfrac{3}{2}\varepsilon^{abcd}\varepsilon_{\alpha\beta\gamma}F^\alpha_{ab}D_AF^\beta_{cc'} \wedge D_AF^\gamma_{dd'} \wedge e^{c'} \wedge e^{d'}$$
$$+ \tfrac{3}{2}\varepsilon^{abcd}\varepsilon_{\alpha\beta\gamma}D_AF^\alpha_{ab} \wedge D_AF^\beta_{cc'} \wedge e^{c'} \wedge F^\gamma_{dd'}e^{d'}. \tag{43}$$

We differentiate each term by parts and use the basic identity

$$D^2_AF^\alpha_{ab} = R_a{}^cF^\alpha_{cb} + R_b{}^cF^\alpha_{ac} + \varepsilon^\alpha{}_{\beta\gamma}F^\beta F^\gamma_{ab}. \tag{44}$$

Neglecting irrelevant terms, throwing away closed forms and making use of gravitational Bianchi identities we finally bring (43) to the form

$$-\tfrac{3}{2}\varepsilon^{abcd}\varepsilon_{\alpha\beta\gamma}F^\alpha_{ab}R_c{}^{a'} \wedge (F^\beta_{a'd}F^\gamma + F^\beta_{a'b'}e^{b'} \wedge F^\gamma_{dd'}e^{d'}). \tag{45}$$

After a few manipulations this can be rewritten as

$$3\mathcal{R}^{ab}F^\alpha_{ac}H_\alpha{}^c{}_b{}^*1 + 3\mathcal{Q}H^\alpha \wedge {}^*F_\alpha. \tag{46}$$

Therefore, summing *(42)* and (46), *we* write the Lagrangian density that determines the RF^3-type direct-curvature Yang-Mills field couplings as follows:

$$L_{RF^3} = 6R_{ab} \wedge H^{ab}_\alpha{}^*F^\alpha + 27\mathcal{R}^{ab}F^\alpha_{ac}H_\alpha{}^c{}_b{}^*1 + 15\mathcal{Q}H^\alpha \wedge {}^*F_\alpha. \tag{47}$$

5. Concluding comments

To conclude we would like to emphasise once again that there seems to be no reason to use solely the Einstein-Hilbert action in higher-dimensional spacetimes to describe the dynamics of gravitational fields. The action (4) given by a linear combination of the non-trivial dimensionally continued Euler forms leads also to field equations that involve at most second-order partial derivatives. To be specific, we have considered in this paper a spacetime of dimension $D = 7$ and discussed the Kaluza-Klein reduction of the dimensionally continued Euler forms assuming the product topology $M_4 \times S^3$. The leading terms in Yang-Mills fields we found in the Kaluza-Klein reduced action coincide with the standard Einstein-Yang-Mills 4-action. We saw that the Kaluza–Klein reduction of the action (4) induces particular Yang-Mills self-interaction terms of the generic types F^3, F^4, F^5 and F^6; as well as terms describing direct coupling of the spacetime curvature to Yang-Mills fields. RF^2-type couplings were already known and discussed. There are terms of type RF^3 and RF^4 which are new. We worked out RF^3 couplings induced by the Kaluza-Klein reduction explicitly. It is known that the dimensional reduction procedure selects a particular subset of all possible invariants of the generic type RF^2 [13]. It would be interesting to see whether the RF^3 terms have a similar structure. The physical consequences of these direct-curvature Yang-Mills field couplings ought to be further studied and understood.

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