

A Study of  
Stationary, Axially Symmetric Space-time Geometries  
Satisfying  
Modified Double Duality Equations  
Using  
the Exterior Calculus Package  $X^T R$  for REDUCE

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**Abstract**

*A stationary, axially symmetric metric is used to reduce the modified double self dual curvature equations to a system of coupled ordinary differential equations. Algebraic manipulations are performed by the  $X^T R$  exterior calculus package for REDUCE.*

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# 1 Introduction

Various types of gravitational theories that generalize Einstein's theory were being discussed during the recent years. A generalization that received a lot of attention was due to Stephenson, Kilmister, Yang and several others.<sup>[1, 2, 3]</sup> In this theory the gravitational action is written in analogy with Yang-Mills type gauge theories, so that the field equations are obtained by varying a quadratic curvature invariant. In a perturbative approach to field quantisation this theory leads to a renormalisable, albeit in general, non-unitary quantum gravity.<sup>[4]</sup> Even at the classical level the theory has some serious problems. Namely, it admits non-physical solutions<sup>[5, 6]</sup>. Therefore, in an attempt to constrain the quadratic theory further it was suggested to add on the Einstein-Hilbert action.<sup>[7]</sup> If it is further allowed to add a cosmological constant, then some remarkable simplifications follow. For a definite value of the cosmological constant, there is a set of modified double dual curvature equations whose integrability conditions give precisely the variational field equations.<sup>[8, 9]</sup> Thus, static spherically symmetric geometries with dynamical torsion are determined by solving the modified double duality equations.<sup>[10]</sup>

We now wish to study stationary, axially symmetric solutions to the modified double duality equations.<sup>[11]</sup> We also start from the Kerr-de Sitter metric, however, we aim in particular to determine geometries that in the limit of vanishing rotation parameter would go to the static, spherically symmetric solutions given in reference<sup>[8]</sup>. Those reduced equations are given below. The long and tedious algebraic manipulations that led to this system of coupled ordinary differential equations are performed using the exterior calculus package X<sup>T</sup>R for REDUCE we developed independently.

## 2 The X<sup>T</sup>R package

Exterior algebra is gaining importance as a computational tool in various branches of theoretical physics. X<sup>T</sup>R is a package which enables computations in this formalism in the algebraic simplification language REDUCE 3<sup>[12]</sup>. The implementation language has been chosen to be REDUCE because of its widespread use and availability. X<sup>T</sup>R has build-in facilities for computations in gravitational and gauge field theories. There already exist some systems which perform computations in the same formalism. May be the most well-acquainted system that exists up to this time is EXCALC<sup>[13]</sup>, which is also a package developed for REDUCE, and now is distributed along with it.

X<sup>T</sup>R is short in code and fast in computation. It uses a different syntax which we believe is more coherent with the general idea behind REDUCE. X<sup>T</sup>R should be regarded as an open-code rather than being a computational black-box. As an example, X<sup>T</sup>R enables declaration of form valued functions at user level and provides a handle to access the form-degree(s) of the argument(s) as well as the ability to declare the way to compute the function's form-degree. Another example, X<sup>T</sup>R avoids the implicit assumption of summation convention over repeated indices; in order to give flexibility to the user for possible coding of efficient algorithms specific to a problem. It is worthwhile to mention the facility build in X<sup>T</sup>R for solving dimensional reduction problems which will not be possible using EXCALC. The technique of dimensional reduction can be described as:

1. There exists an  $n$ -dimensional manifold  $M_n$  with a topology  $M_{n_1} \times M_{n_2}$ .
2. The physical quantities (*like connection 1-forms, curvature 2-forms, the curvature scalar, etc.*) are calculated for  $M_n$ .
3. Similar physical quantities are calculated independently for  $M_{n_1}$  and  $M_{n_2}$ .
4. Using those results the physical quantities for  $M_n$  are tried to be expressed in terms of the physical quantities of  $M_{n_1}$  and  $M_{n_2}$  only.

The underlying topology of the manifold reflects in the coordinate chart  $\mathbf{x}^M$  that it is partitioned as  $(x^\mu, y^m)$  where the indices run as  $\mu=0,1,\dots,n_1-1$  and  $m=n_1,\dots,n$ . This idea of index separation is easily handled with the facilities  $X^{\text{TR}}$  provides. It is possible to define the dimensional reduction spaces, declare forms to exist in any of them, or to have contributions to both of the spaces. Furthermore, it is also possible to declare indexed forms to have a lifestyle according to their symbolic indices. As it would be expected, facilities, that define the way the indices shall split while the dimensional reduction is carried out, are present.

A manual, which will be provided upon request, gives a full description of  $X^{\text{TR}}$  assuming the user has a very limited knowledge of REDUCE and automated symbolic computation. Detailed information on the inner structure of the package which will aid the advanced user for possible extensions, is also provided.

It is very common that the operations performed in exterior calculus fall into one of the two classes, namely the one in which the operations are antiderivations, therefore the distribution rule is:

$$\Delta(q \wedge r) = \Delta q \wedge r + (-)^{\text{deg}(q)} q \wedge \Delta r$$

and the one in which the operations are derivations, so the distribution rule is:

$$\square(q \wedge r) = \square q \wedge r + q \wedge \square r$$

$X^{\text{TR}}$  has build in facilities to declare operators with these distribution rules. Also it is possible to have multilinear operations.

Among the capabilities of the package are:

- Setting the dimensions and signature of the form space.
- Declaration of variables and REDUCE operators to be forms of any degree.
- Performing the basic operations of exterior calculus, namely:

- **Exterior (wedge) product**

$X^{\text{TR}}$  knows the followings:

- \* The commutation rule,  $p \wedge q = (-)^{\text{deg}(p) \text{deg}(q)} \cdot q \wedge p$
- \* A wedge product which contains a form in two different places will cause the product to be zero if this is an odd-degree-form.
- \* Any wedge product that sums up to a higher degree than the space dimension in degrees is zero.
- \* The wedge product of two zero forms is the ordinary multiplication of them.

- **Exterior differentiation**

- \* It acts as an antiderivative type of operator over the wedge product.
- \* Is nilpotent.
- \* Acts as the ordinary differentiation if applied to zero-forms.
- \* It is possible to control the expansion into partial derivatives if it is a function that is subject to the operation.

- **Interior product**

- \* It acts as an antiderivative type of operator over the wedge product.
- \* Is nilpotent.
- \* It is possible to force the system to leave the interior products, which can not be explicitly calculated, as they are.

- **Hodge duality**

- \* If the vierbein and the signature is provided then the system is able to carry out the substitution for the hodge operation.
- \* It is known how multi hodes simplify.

In addition to these capabilities  $X^{\text{TR}}$  is also able to carry out the calculations in an orthonormal base, if it is provided with the vierbein/vielbein. The user has also control over the exterior operations by setting some flags of the package ON or OFF.

It is worth to mention about the facility build in  $X^{\text{TR}}$  for “Dimensional Reduction”.

### 3 Exterior Calculus: Notation and Conventions

The modified double duality equations are

$$*(R^{ab} + \frac{\lambda}{2} e^a \wedge e^b) = -(R^{ab} + \frac{\lambda}{2} e^a \wedge e^b)^* \quad (1)$$

where  $\lambda$  is an arbitrary real parameter.  $(e^a)$  are the orthonormal coframes in terms of which the metric

$$g = \eta_{ab} e^a \otimes e^b \quad (2)$$

We take  $\eta_{ab} = \text{diag}(- + + +)$ . Together with the connection 1-forms  $(\omega^a_b)$  the coframes satisfy the structure equations

$$de^a + \omega^a_b \wedge e^b = T^a \quad (3)$$

$$d\omega^a_b + \omega^a_c \wedge \omega^c_b = R^a_b \quad (4)$$

Here  $T^a = T_{bc}^a e^b \wedge e^c$  are the torsion 2-forms and  $R^a_b = \frac{1}{2} R_{cd}^a e^c \wedge e^d$  are the curvature 2-forms of space-time. A  $*$  to the left of a form describes its Hodge dual, defined so that the invariant volume element

$$*1 = e^0 \wedge e^1 \wedge e^2 \wedge e^3 = \frac{1}{4!} \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \quad (5)$$

A  $*$  to the right of a 2<sup>nd</sup> rank antisymmetric tensor denotes its dual. For instance,

$$R_{ab}^* = \frac{1}{2!} \epsilon_{ab}^{cd} R_{cd} \quad (6)$$

The modified double duality equations may also be written in the form

$$*R_{ab} + R_{ab}^* = \lambda *(e_a \wedge e_b) \quad (7)$$

### 4 Stationary, axisymmetric double dual curvatures

We consider solutions described by the Kerr de Sitter metric <sup>[14]</sup>

$$g = -\frac{\delta_r}{r^2 + a^2 \cos^2 \theta} \left[ \frac{dt - a \sin^2 \theta d\varphi}{1 - \frac{1}{3}ka^2} \right]^2 + (r^2 + a^2 \cos^2 \theta) \left\{ \frac{dr^2}{\delta_r} + \frac{d\theta^2}{1 - \frac{1}{3}ka^2 \cos^2 \theta} \right\} + \sin^2 \theta \left( \frac{1 - \frac{1}{3}ka^2 \cos^2 \theta}{r^2 + a^2 \cos^2 \theta} \right) \left[ \frac{adt - (r^2 + a^2)d\varphi}{1 - \frac{1}{3}ka^2} \right]^2 \quad (8)$$

expressed in terms of Boyer-Lindquist coordinates  $(t, r, \theta, \varphi)$ . <sup>[15]</sup> We define the following for later convenience:

$$\begin{aligned} (Z_1)^2 &\triangleq 1 - \frac{k}{3}a^2 \\ (Z_2)^2 &\triangleq r^2 + a^2\mu^2 \\ (Z_3)^2 &\triangleq 1 - \frac{k}{3}a^2\mu^2 \\ M^2 &\triangleq 1 - \mu^2 \\ (\delta_r)^2 &\triangleq \frac{k}{3}(r^4 + a^2r^2) + r^2 - 2mr + a^2 \end{aligned}$$

where  $\mu = \cos \theta$ .  $m$  shows the Schwarzschild mass,  $a$  the rotation, and  $k$  is a real parameter. We take the orthonormal co-frames

$$\begin{aligned} e^0 &= \frac{\delta_r}{(Z_1)^2 Z_2} dt - \frac{aM^2 \delta_r}{(Z_1)^2 Z_2} d\varphi \\ e^1 &= \frac{aM Z_3}{(Z_1)^2 Z_2} dt - \frac{M Z_3 (a^2 + r^2)}{(Z_1)^2 Z_2} d\varphi \\ e^2 &= \frac{Z_2}{\delta_r} dr \\ e^3 &= \frac{Z_2}{M Z_3} d\mu \end{aligned} \quad (9)$$

Then the Levi-Civita connection 1-forms are found as follows:

$$\begin{aligned} \dot{\omega}^0_1 &= \frac{arM Z_3}{(Z_2)^3} e^2 + \frac{a\mu\delta_r}{(Z_2)^3} e^3 \\ \dot{\omega}^0_2 &= \frac{1}{(Z_2)^5 \delta_r} [a^2 \mu^2 (3a^2 \mu^2 \delta_r \frac{\partial \delta_r}{\partial r} + ka^2 r^3 - 3a^2 r + 3kr^5 + 3r^3) + 3mr^4 - 3a^2 r^3 + kr^7] e^0 \\ &\quad + \frac{3M Z_1 ar}{(Z_2)^3} e^1 \\ \dot{\omega}^0_3 &= \frac{a^2 \mu M Z_3}{(Z_2)^3} e^0 - \frac{a\mu\delta_r}{(Z_2)^3} e^1 \\ \dot{\omega}^1_2 &= -\frac{arM Z_3}{(Z_2)^3} e^0 + \frac{r\delta_r}{(Z_2)^3} e^1 \\ \dot{\omega}^1_3 &= \frac{a\mu\delta_r}{(Z_2)^3} e^0 + \frac{\mu}{3M Z_3 (Z_2)^3} [ka^2 (\mu^2 (Z_2)^2 - M^2 r^2) - 3(a^2 + r^2)] e^1 \\ \dot{\omega}^2_3 &= \frac{a^2 \mu M Z_3}{(Z_2)^3} e^2 - \frac{r\delta_r}{(Z_2)^3} e^3 \end{aligned} \quad (10)$$

We will construct solutions such that the full connection 1-forms are of the following form:

$$\begin{aligned} \omega_{01} &= \Omega_1 e^2 + \Omega_2 e^3 \\ \omega_{02} &= \Omega_3 e^0 + \Omega_4 e^1 \\ \omega_{03} &= \Omega_5 e^0 + \Omega_6 e^1 \\ \omega_{12} &= \Omega_7 e^0 + \Omega_8 e^1 \\ \omega_{13} &= \Omega_9 e^0 + \Omega_{10} e^1 \\ \omega_{23} &= \Omega_{11} e^2 + \Omega_{12} e^3 \end{aligned} \quad (11)$$

After manipulating the structure equations we get:

$$\begin{aligned} *R_{01} + R_{01}^* &= S_1 e^0 \wedge e^1 + S_2 e^2 \wedge e^3 \\ *R_{02} + R_{02}^* &= S_3 e^0 \wedge e^2 + S_4 e^0 \wedge e^3 + S_5 e^1 \wedge e^2 + S_6 e^1 \wedge e^3 \\ *R_{03} + R_{03}^* &= S_7 e^0 \wedge e^2 + S_8 e^0 \wedge e^3 + S_9 e^1 \wedge e^2 + S_{10} e^1 \wedge e^3 \\ *R_{12} + R_{12}^* &= -S_{10} e^0 \wedge e^2 + S_9 e^0 \wedge e^3 - S_8 e^1 \wedge e^2 + S_7 e^1 \wedge e^3 \\ *R_{13} + R_{13}^* &= S_6 e^0 \wedge e^2 - S_5 e^0 \wedge e^3 + S_4 e^1 \wedge e^2 - S_3 e^1 \wedge e^3 \\ *R_{23} + R_{23}^* &= S_2 e^0 \wedge e^1 - S_1 e^2 \wedge e^3 \end{aligned} \quad (12)$$

The functions  $S_1, \dots, S_{10}$  will be given explicitly below. In terms of these functions the modified double duality equations (7) read

$$S_2 = -S_6 = S_9 = \lambda \quad (13)$$

$$S_1 = S_3 = S_4 = S_5 = S_7 = S_8 = S_{10} = 0$$

These functions classify into two distinct sets. Six of them, namely  $S_1, S_2, S_4, S_5, S_8, S_9$  are of the generic form:

$$\frac{1}{(Z_2)^3} \left\{ M Z_3 (a^2 \mu + (Z_2)^2 \partial_\mu) \Omega_A + \delta_r (r + (Z_2)^2 \partial_r) \Omega_B + (Z_2)^3 [\Omega_C \Omega_D + \Omega_E \Omega_F + \Omega_G \Omega_H + \Omega_I \Omega_J] \right\}$$

The actual expressions we give as a table:

	A	B	C·D	E·F	G·H	I·J
$S_1$	-1	2	8 · 9	-7 · 10	6 · 3	-5 · 4
$S_2$	-11	12	8 · 3	-7 · 4	-6 · 9	5 · 10
$S_4$	9	-4	-8 · 1	-7 · 12	-6 · 11	5 · 2
$S_5$	3	10	-8 · 11	7 · 2	6 · 1	5 · 12
$S_8$	-7	-6	4 · 11	-3 · 2	-1 · 10	-12 · 9
$S_9$	5	-8	-4 · 1	-3 · 12	2 · 9	-11 · 10

Table 1

The first horizontal line in the table means:

$$S_1 = \frac{1}{(Z_2)^3} \left\{ -M Z_3 (a^2 \mu + (Z_2)^2 \partial_\mu) \Omega_1 + \delta_r (r + (Z_2)^2 \partial_r) \Omega_2 + (Z_2)^3 [\Omega_8 \Omega_9 - \Omega_7 \Omega_{10} + \Omega_6 \Omega_3 - \Omega_5 \Omega_4] \right\}$$

The other expressions can be similarly written out.

The second set of functions  $S_3, S_6, S_7, S_{10}$  are of the generic form

$$\frac{1}{(Z_2)^3 Z_3} \left\{ M Z_3 [-r \delta_r + (Z_2)^2 \partial_r (\delta_r) + (Z_2)^2 \delta_r \partial_r] \Omega_B + \right. \\ \left. [\mu (\frac{1}{3} k r^2 a^2 (2\mu^2 - 1) - (r^2 + a^2) + \frac{1}{3} k a^4 \mu^4) + M^2 (Z_3)^2 (Z_2)^2 \partial_\mu] \Omega_A + \right. \\ \left. + 2a M Z_3 [\mu \delta_r \Omega_C + r M Z_3 \Omega_D] + (Z_2)^3 Z_3 [\Omega_E \Omega_F + \Omega_G \Omega_H + \Omega_I \Omega_J + \Omega_K \Omega_L] \right\}$$

The actual expressions are read from the following table:

	A	B	C	D	E·F	G·H	I·J	K·L
$S_3$	4	9	-3	-10	8 · 2	-7 · 11	6 · 12	5 · 1
$S_6$	10	-3	-9	4	-8 · 12	-7 · 1	6 · 2	-5 · 11
$S_7$	6	-7	-5	8	-4 · 12	-3 · 1	2 · 10	-11 · 9
$S_{10}$	-8	-5	7	6	-4 · 2	3 · 11	-1 · 9	-12 · 10

Table 2

Their solutions will be discussed elsewhere.<sup>[16]</sup>

## 5 Conclusion

The new developed exterior calculus package, X<sup>T</sup>R, for REDUCE is introduced. Besides its open code design philosophy which enables flexibility in special applications, this package provides additional computational abilities for the dimensional reduction technique that makes it unique. In the future the package will be further developed to include

- Ability to handle Lie algebra valued forms.
- Automatic simplification of indexed quantities.

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## APPENDIX

X<sup>T</sup>R extended REDUCE program for the problem described in the preceding sections.

```

% Kerr-de Sitter metric in Boyer-Lindquist coordinates ver 2.1;

COORDINATE T1,PHI,R,MU;
OPERATOR T2,T3,M,DELR;
OPERATOR CONNECTION, DCONNECTION;
ANTISYMMETRIC CONNECTION, DCONNECTION;

% T1 has no R,MU dependency;

LET T2 = T2(R,MU),
    T3 = T3(MU),
    M = M(MU),
    DELMU = T3(MU)*M(MU),
    DELR = DELR(R);

LET DF(T2(R,MU),R) = R/T2,
    DF(T2(R,MU),MU) = AA**2*MU/T2,
    DF(T3(MU),MU) = (T1**2-1)*MU/T3,
    DF(M(MU),MU) = -MU/M,
    DF(DELR(R),R) = (2/3*K*R**3 + (2-T1**2)*R - MASS)/DELR;

INTEGER PROCEDURE ETA(I); %Used for index raising and lowering
    IF I=0 THEN -1 ELSE 1;

PROCEDURE W(A,B)$
    1/2*(ETA(B)*(A **. D E B) +
    ETA(A)*(B **. D E A) +
    FOR C:= 0 : 3 SUM
    (A **. (B **. D E C)*ETA C *E C)$
    Definition of how to obtain the
    connection one-forms.

MATRIX VIERBEIN(4,4), INVIERBEIN(4,4);

VIERBEIN(1,1) := DELR/T1**2/T2 $
VIERBEIN(1,2) := -AA*M**2*DELR/T1**2/T2 $
VIERBEIN(2,1) := AA*M*T3/T1**2/T2 $
VIERBEIN(2,2) := -M*(R**2+AA**2)*T3/T1**2/T2 $
VIERBEIN(3,3) := T2/DELR $
VIERBEIN(4,4) := T2/DELMU $

% Below we define the inverse vierbein, infact, this could be left ;
% to the system by calling GENERATE as GENERATE(T) but the result ;
% is not as compact as the one below due to the matrix inversion ;
% routines of REDUCE itself ;

INVIERBEIN(1,1) := T1**2*(R**2+AA**2)/T2/DELR $
INVIERBEIN(1,2) := -T1**2*AA*M/T2/T3 $
INVIERBEIN(2,1) := T1**2*AA/T2*DELR $
INVIERBEIN(2,2) := -T1**2/T2/T3/M $
INVIERBEIN(3,3) := DELR/T2 $
INVIERBEIN(4,4) := DELMU/T2 $

ON DEREXP;

GENERATE();

ON INBASE; % We want the results in terms of the orthonormal coframes ;

FACTOR E O&E 1&E 2, E O&E 1&E 3, E O&E 2&E 3, E 1&E 2&E 3,
    E O&E 1, E O&E 2, E O&E 3, E 1&E 2, E 1&E 3, E 2&E 3,
    E 0, E 1, E 2, E 3;

OFF UNKINPRD$ % No unknown interior product shall remain;

```



```

% First compute the connections and store the result into CONNECTION;
FOR A := 0 : 3 DO
  FOR B:= A+1 : 3 DO
    CONNECTION(A,B) := W(A,B)$
% Now compute the D's of the connections;
FOR A := 0 : 3 DO
  FOR B := A+1 : 3 DO
    DCONNECTION(A,B) := D W(A,B)$
% Compute the riemann 2-formS ;
OPERATOR RIEMANN;
ANTISYMMETRIC RIEMANN;
FOR A := 0 : 3 DO
  FOR B := A+1 : 3 DO
    RIEMANN(A,B) := DCONNECTION(A,B) +
      FOR C := 0 : 3 SUM
        ETA(C)*CONNECTION(A,C) & CONNECTION(C,B);
% PROMETEUS will contain The modified double duality equations;
FOR A := 0 : 3 DO
  FOR B := A+1 : 3 DO
    PROMETEUS(A,B) := RHODGE RIEMANN(A,B) + # RIEMANN(A,B);
RECOORDINATE TI,PHI,R,MU; % Just to speed up;
LET
  T1**2      = 1 - K/3*AA**2,
  T2(R,MU)**2 = R**2 + AA**2*MU**2,
  T3(MU)**2  = 1-K/3*AA**2*MU**2,
  M(MU)**2   = 1-MU**2,
  DELR(R)**2 = K/3*(R**4+AA**2*R**2)+R**2-2*MASS*R+AA**2;
% The outputting of the results;
FOR A := 0 : 3 DO FOR B := A+1 : 3 DO
  CONNECTION(A,B) := CONNECTION(A,B);
FOR A := 0 : 3 DO FOR B := A+1 : 3 DO
  DCONNECTION(A,B) := DCONNECTION(A,B);
FOR A := 0 : 3 DO FOR B := A+1 : 3 DO
  RIEMANN(A,B) := RIEMANN(A,B);
FOR A := 0 : 3 DO FOR B := A+1 : 3 DO
  PROMETEUS(A,B) := PROMETEUS(A,B);

```