CENG477 Recitation 1 - Math Review

Ahmet Oğuz Akyüz
LaTeX by Kadir Cenk Alpay

October 12, 2018

Introduction

In Computer Graphics (CG), points, vectors, and matrices are used extensively.

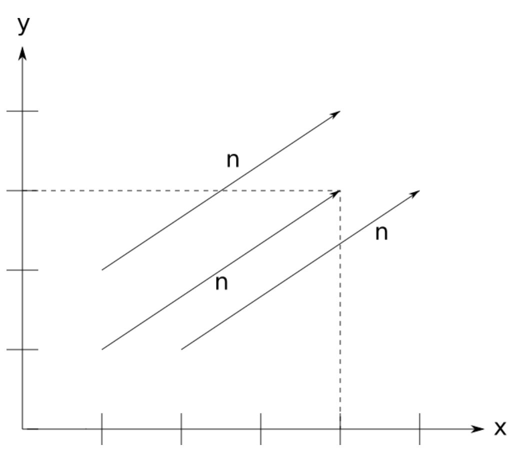
- Points are used to represent object coordinates.
- Vectors are used to represent light propagation, orientation of surfaces, direction of light sources and cameras.
- Matrices are used to represent object transformations.

Vectors

- In CG, we deal with 2D, 3D and 4D vectors.

E.g: a 2D vector in column notation: $n = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

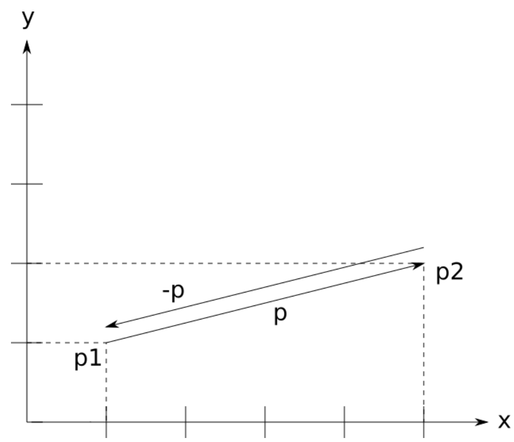
The same 2D vector in row notation: $n = \begin{bmatrix} 3 & 2 \end{bmatrix}^T$



- All three vectors are the same vector n.

- That is, a vector does not have a unique position in space.

- A vector can be seen as a line segment that connects two points.



$$p = p_2 - p_1$$
$$-p = p_1 - p_2$$

Vector Properties

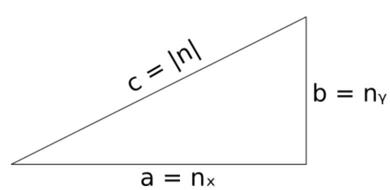
- Each vector has two fundamental properties: direction and magnitude.

- If $n = \begin{bmatrix} n_x \\ n_y \end{bmatrix}$, then $|n| = \sqrt{n_x^2 + n_y^2}$

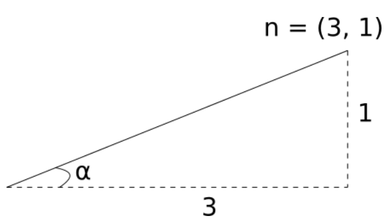
- Remember the Pythagorean theorem:

$$c^2 = a^2 + b^2$$

$$c = \sqrt{a^2 + b^2}$$



- Direction is determined by the relative relationship of a vector's components:



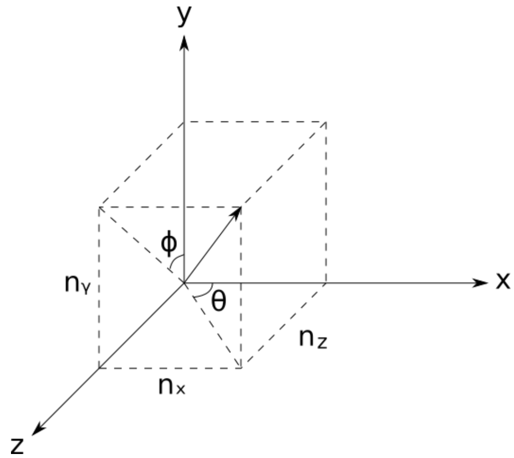
$tan(\alpha) = \frac{1}{3}$ $\quad \alpha = tan^{-1}(\frac{1}{3})$

or

$\alpha = cos^{-1}(\frac{3}{|n|})$ $\quad \alpha = sin^{-1}(\frac{1}{|n|})$

- Two vectors with the same direction and magnitude are the same vectors.

3D Vectors



$$n = \begin{bmatrix} n_x & n_y & n_z \end{bmatrix}^T \qquad |n| = \sqrt{n_x^2 + n_y^2 + n_z^2}$$

- For the direction, we now have two angles (θ and ϕ).

θ is the angle on the **x - z** plane.

ϕ is the angle on the **y - z** plane.

$$\theta = tan^{-1}(n_z/n_x)$$

$$\phi = tan^{-1}(n_z/n_y)$$

Vector Operations

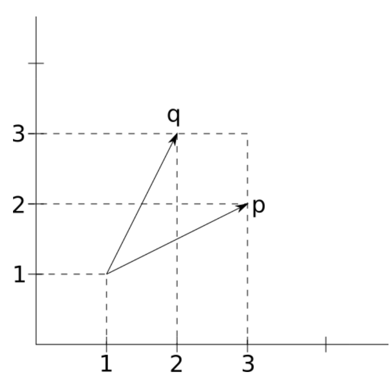
- **Scalar multiplication**: $n = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$, $2n = \begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix}$, $kn = \begin{bmatrix} kn_x \\ kn_y \\ kn_z \end{bmatrix}$

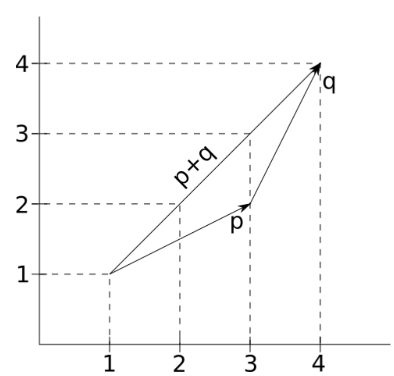
  - Scalar multiplication only changes the magnitude, but not the direction.

- **Addition and subtraction**: $p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$, $q = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}$, $p + q = \begin{bmatrix} p_x + q_x \\ p_y + q_y \\ p_z + q_z \end{bmatrix}$

  $p + q = q + p$ (commutativity)

  - But note that $p - q \neq q - p$





Unit Vectors

- A unit vector has a magnitude of 1.

E.g. $i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $|i| = 1$

- Converting a vector into a unit form is called <u>normalizing</u> and is achieved by dividing a vector's components by its magnitude.

E.g. $r = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then $|r| = \sqrt{x^2 + y^2 + z^2}$ and $\hat{r} = \frac{1}{|r|}.r$ (Note that $|\hat{r}| = 1$)

Cartesian Vectors

- Cartesian vectors are unit vectors that are aligned with the axes of the Cartesian coordinate system.

$$i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Every 3D vector can be represented with a linear combination of **i**, **j**, and **k**.

E.g. $r = a.i + b.j + c.k$, this is the same as $r = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

E.g:

$$r = ai + bj + ck$$
$$s = di + ej + fk$$
$$r \pm s = (a \pm d)i + (b \pm e)j + (c \pm f)k$$

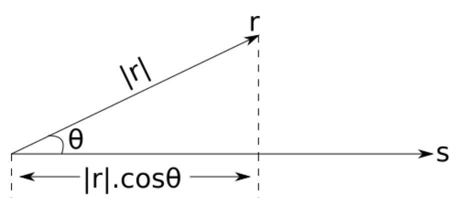
Vector Multiplication

Scalar (Dot) Product

- If we have two vectors **r** and **s**, the dot product is equal to: $r.s = |r||s|cos(\theta)$

where the θ is the angle between the vectors.

- Dot product is the multiplication of the magnitudes of two vectors that are projected onto each other.



Ex:

$$r = ai + bj + ck$$
$$s = di + ej + fk$$

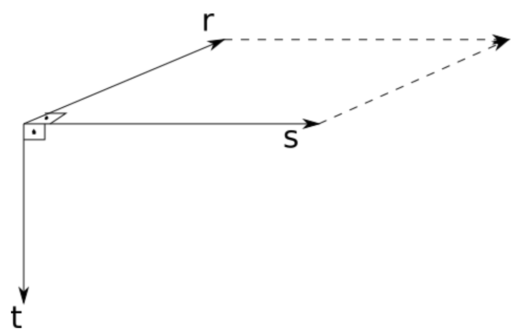
$$\begin{aligned} r \cdot s &= (ai + bj + ck) \cdot (di + ej + fk) \\ &= ai \cdot (di + ej + fk) + bj \cdot (di + ej + fk) + ck \cdot (di + ej + fk) \\ &= ad(i \cdot i) + \cancel{ae(i.j)} + \cancel{af(i.k)} + \cancel{bd(j.i)} + be(j \cdot j) + \cancel{bf(j.k)} + \cancel{cd(k.i)} + \cancel{ce(k.j)} + cf(k \cdot k) \\ &= ad + be + cf = |r||s|cos\theta \end{aligned}$$

What is i · i , j · j , k · k ? = 1
What is i · j , i · k , j · k ? = 0

- Note that the dot product of two vectors gives a <u>scalar</u>. Also note that $r.s = s.r$

Vector (Cross) Product

- Cross product of two vectors gives another vector; and this new vector is perpendicular to both vectors.



$$r \times s = t \quad \text{such that} \quad |t| = |r||s|sin\theta$$

- In other words, **t** is perpendicular to the plane created by **r** and **s**.

Ex:

$$r = ai + bj + ck$$
$$s = di + ej + fk$$

$$\begin{aligned} r \times s &= (ai + bj + ck) \times (di + ej + fk) \\ &= ai \times (di + ej + fk) + bj \times (di + ej + fk) + ck \times (di + ej + fk) \\ &= \cancel{ad(i \times i)} + ae(i \times j) + af(i \times k) + bd(j \times i) + \cancel{be(j \times j)} + bf(j \times k) + cd(k \times i) + ce(k \times j) + \cancel{cf(k \times k)} \end{aligned}$$

i×i = 0 , j×j = 0 , k×k = 0
i×j = k , i×k = -j , j×k = i (assuming right hand rule)

$$\begin{aligned} &= aek - afj - bdk + bfi + cdj - cei \\ &= (bf - ce)i + (cd - af)j + (ae - bd)k \end{aligned}$$

- This result can be remembered using determinants.

- Note that $r \times s \neq s \times r$

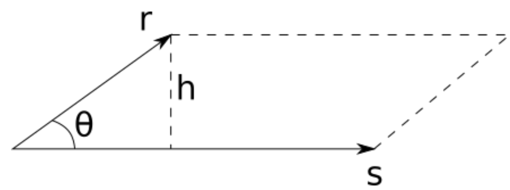
Physical Meaning of Cross Product

$r \times s = t$ where $|t| = |r||s|sin\theta$

- Area of the parallelogram formed by r and s:

$$A = h.|s| = |r||s|sin\theta$$

- Thus the magnitude of the cross product is equal to the area of the parallelogram.



$|h| = |r|sin\theta$

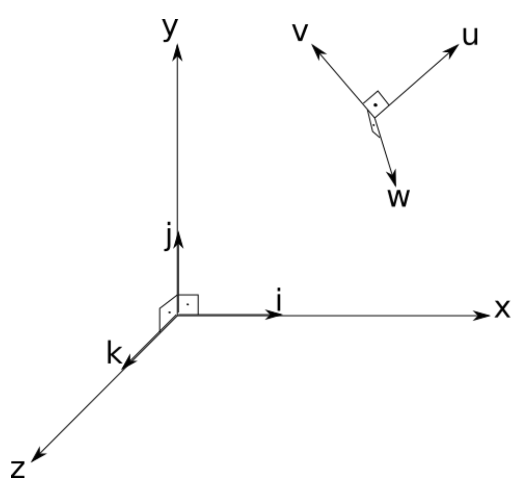
Orthonormal Bases and Coordinate Systems

- Managing coordinate systems is one of the most important tasks of a CG program.
- Each object can be defined in its own coordinate system.
- Cameras, lights may be defined in other CS.
- Yet, everything has to work together somehow.

- An orthonormal basis is a basis made up of three perpendicular (ortho) unit (normal) vectors.

- Let's call these vectors u, v, w.

- This basis is right-handed provided that $w = u \times v$.



- Note that Cartesian vectors (i, j, k) form just one of infinitely many possible orthonormal basis.

- We call it the canonical or global basis.

- u, v, w vectors form a local basis.

- Note that u, v, w are defined in terms of i, j, k.

E.g. $u = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$u = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j \quad v = -\frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j \quad w = k$

- Let's say vector a is stored in terms of the u, v, w CS. How can we find its coordinates in the canonical CS?

$$a = \begin{bmatrix} a_u \\ a_v \\ a_w \end{bmatrix}, \quad \text{what is} \begin{bmatrix} a_i \\ a_j \\ a_k \end{bmatrix} ?$$

- Because u, v, w themselves are stored in the canonical CS, the expression $a_u.u + a_v.v + a_w.w$ already gives a result in canonical CS.

E.g. Given $a = \begin{bmatrix} 1 \\ 2 \\ 0.5 \end{bmatrix}$ in u, v, w defined as $u = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ $w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

What are the coordinates of a in the global (canonical) XYZ coordinate system?

$$1 \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} + 0.5 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$$

- Now how can we do the opposite? How can we find the u, v, w coordinates of vector b stored in canonical CS?

For some u_b, v_b, w_b we know that $u_b u + v_b v + w_b w = b$

Take the dot product of both sides: $(u_b u + v_b v + w_b w) \cdot u = b \cdot u$

$u_b = b \cdot u$ that is, to get u_b take the dot product of u with b.

So, $b_{u,v,w} = \begin{bmatrix} b \cdot u \\ b \cdot v \\ b \cdot w \end{bmatrix}$ (e.g reverse the example above)

Constructing a Basis from a Single Vector

- Given a, we want w to point in the same direction as a.

$w = \frac{a}{|a|}$ $\quad u = \frac{t \times w}{|t \times w|}$ $\quad v = w \times u$

Matrices

- Matrices are especially used for transformations in CG. So, we need to know their properties.

- A matrix is an array of numeric elements that follow certain arithmetic rules.

E.g. $A = \begin{bmatrix} 1.7 & -1.2 & 4.2 \\ 3.0 & 4.5 & -7.2 \end{bmatrix}$ $\quad A$ is a 2 by 3 matrix.

Matrix Arithmetic

Multiplication by a Scalar: $2 \cdot \begin{bmatrix} 1 & -4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -8 \\ 6 & 4 \end{bmatrix}$, $k \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$

Matrix Addition: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$

Matrix Multiplication: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$

Two matrices can only be multiplied if the number of columns of the left matrix is equal to the number of rows of the right matrix:

$$A_{m \times n} \, . \, B_{n \times k} \; = \; C_{m \times k}$$

Matrix-Vector Multiplication: We can think of an n dimensional vector as an $n \times 1$ matrix:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax+by+cz \\ dx+ey+fz \\ gx+hy+iz \end{bmatrix}$$

You can think of this as:

$$x \begin{bmatrix} a \\ d \\ g \end{bmatrix} + y \begin{bmatrix} b \\ e \\ h \end{bmatrix} + z \begin{bmatrix} c \\ f \\ i \end{bmatrix}$$

Identity Matrix: An identity matrix is a **square matrix** whose elements are all zeros except the diagonal elements which are all 1.

$$I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix Inverse: The inverse of a matrix A is denoted as A^{-1} and it has the following property:

$$A.A^{-1} = I$$

Note that only square matrices have inverses.

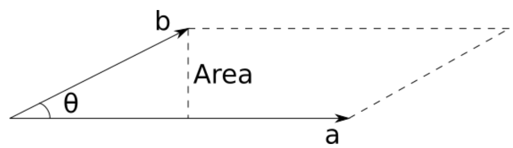
Also note that $(AB)^{-1} = B^{-1}A^{-1}$. How about $(ABC)^{-1}$?

Matrix Transpose: Transposing a matrix changes the columns with the rows:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \qquad (AB)^T = B^T A^T \;\; \text{(similar to the inversion)}$$

The Determinants

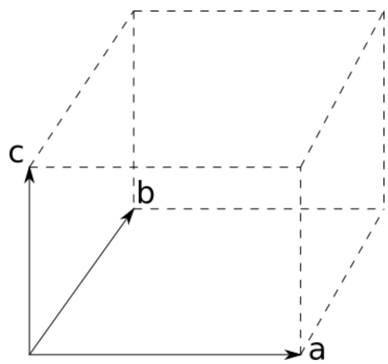
The determinant of two 2-D vectors is equal to the signed area of the parallelogram formed by these vectors.



$$Area = |a|.|b|.sin\theta$$

$$M = \begin{bmatrix} a_x & b_x \\ a_y & b_y \end{bmatrix} \quad , \quad |M| = Area = a_x b_y - a_y b_x$$

In 3D, the determinant of three 3-D vectors is equal to the signed volume of the parallelepiped defined by the vectors.



$$M = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix}$$

$$\begin{aligned} |M| = Volume = \ & a_x(b_y c_z - b_z c_y) - \\ & b_x(a_y c_z - a_z c_y) + \\ & c_x(a_y b_z - a_z b_y) \end{aligned}$$

There is a trick for computing cross-product using the determinant:

$r = ai + bj + ck$
$s = di + ej + fk$

$r \times s$:

$$\begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix} = i\ (bf - ce) - j\ (af - cd) + k\ (ae - bd) = i\ (bf - ce) + j\ (cd - af) + k\ (ae - bd)$$