CENG 222

Statistical Methods for Computer Engineering

Week 2

Chapter 3

Discrete Random Variables and Their Distributions

Random Variables

- A random variable (r.v.) associates a unique numerical value with each outcome in the sample space. It is a real (or integer)-valued function from a sample space Ω into real (or integer) numbers.
- Similar to events it is denoted by an uppercase letter (e.g., X or Y) and a particular value taken by a r.v. is denoted by the corresponding lowercase letter (e.g., x or y).

Examples

- Toss three coins. X = number of heads
- Pick a student from the Computer Engineering Department.

  X = age of the student
- Observe lifetime of a light bulb

  X = lifetime in minutes
- X may be discrete or continuous

Random variables and probabilities

- Consider the three coin tosses example where X = number of heads
- Each value of a random variable is an event
  - E.g. $\boldsymbol{P}(X = 2)$
- We may compute probabilities of these events:
  - $\boldsymbol{P}(X = 2) = \boldsymbol{P}(\mathrm{HHT})+\boldsymbol{P}(\mathrm{HTH})+\boldsymbol{P}(\mathrm{THH}) = 3/8$

Discrete random variables

- If the values a r.v. can take is finite or countably infinite then the r.v. is discrete
- Suppose that we can calculate $\boldsymbol{P}(X=x)$ for every value of x. The collection of these probabilities can be viewed as a function of X. The probability mass function (p.m.f) of a discrete r.v. is given by

$$f_X(x) = P(X = x)$$

Distribution

- Collection of all probabilities related to X is the distribution of X.
- $f_X(x) = \boldsymbol{P}(X = x)$ is the probability mass function (pmf)
- Cumulative distribution function (cdf)
  - $F(x) = \boldsymbol{P}(X \leq x) = \sum_{y \leq x} f_X(y)$

- The set of all possible values of X is called the **support** of the distribution F.

Continuous random variables

- A r.v. X is continuous if it can take any value from one or more intervals of real numbers.
- We cannot use p.m.f because $P(X=x) = 0$ since there are infinitely many possible outcomes. Instead we use a *probability density function* (p.d.f), $f_X(x)$, such that the areas under the curve represent probabilities

Continuous random variable

- The p.d.f ($f_X(x)$) of a continuous r.v. X is the function that satisfies

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt \quad \text{for all } x$$

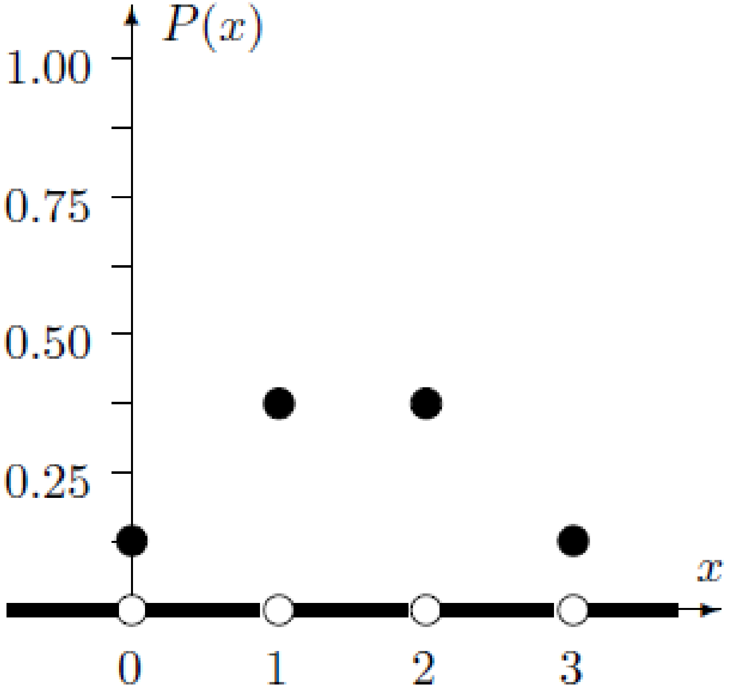
where $F_X(x)$ is the cumulative distribution function (c.d.f) of a r.v. X defined by

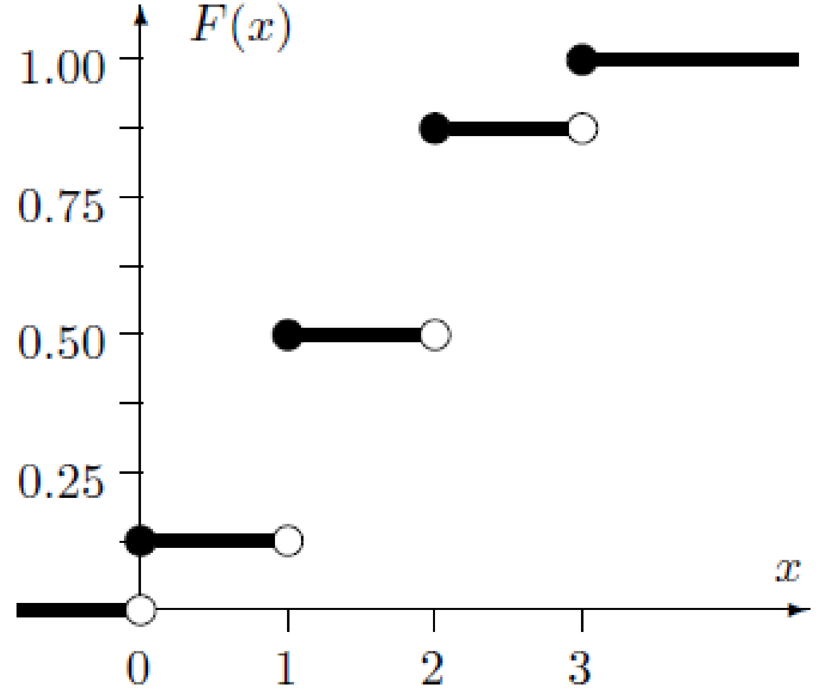
$$F_X(x) = P_X(X \le x) \quad \text{for all } x$$

We will discuss continuous random variables in detail in Chapter 4

pmf and cdf

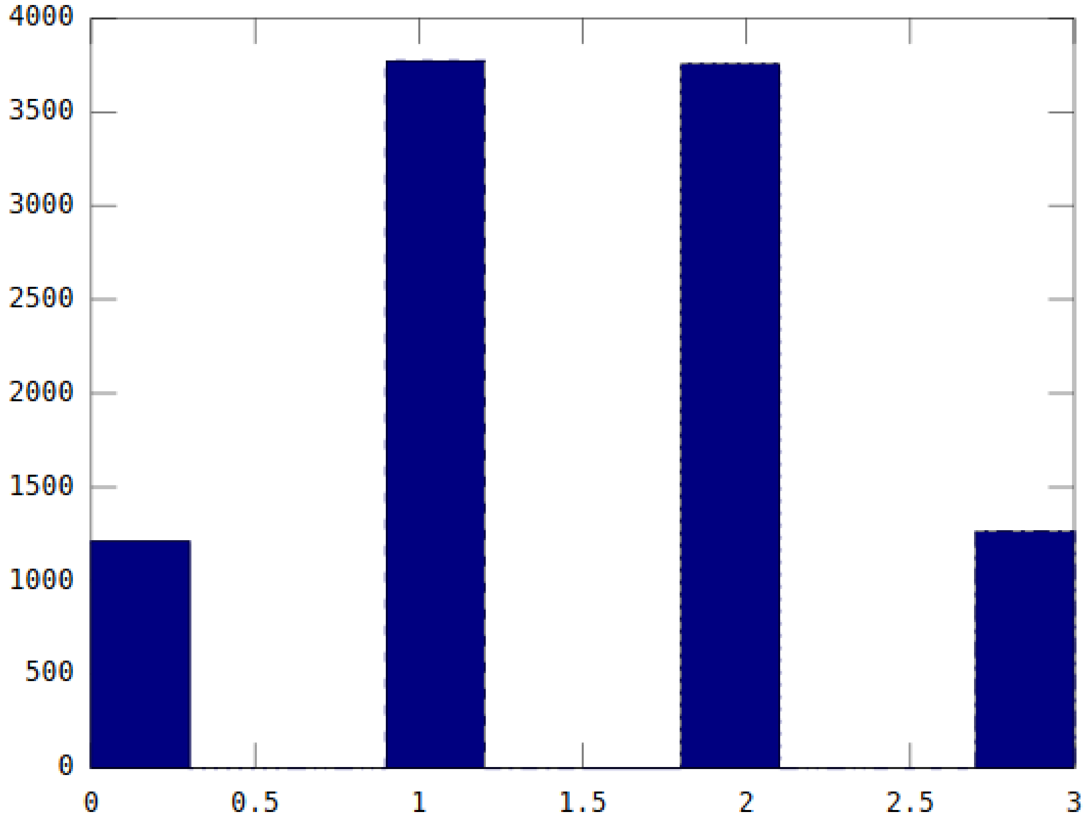
Back to toss of
3 coins





Octave Online

```
octave:1> N = 10000;
octave:2> U = rand(3,N);
octave:3> Y = (U<0.5);
octave:4> X = sum(Y);
octave:5> hist(X);
```



Example 3.3: Errors in two modules

x	$P_1(x)$	$P_2(x)$
0	0.5	0.7
1	0.3	0.2
2	0.1	0.1
3	0.1	0

- Y = total number of errors
- What is the pmf and cdf of Y?

Random Vectors

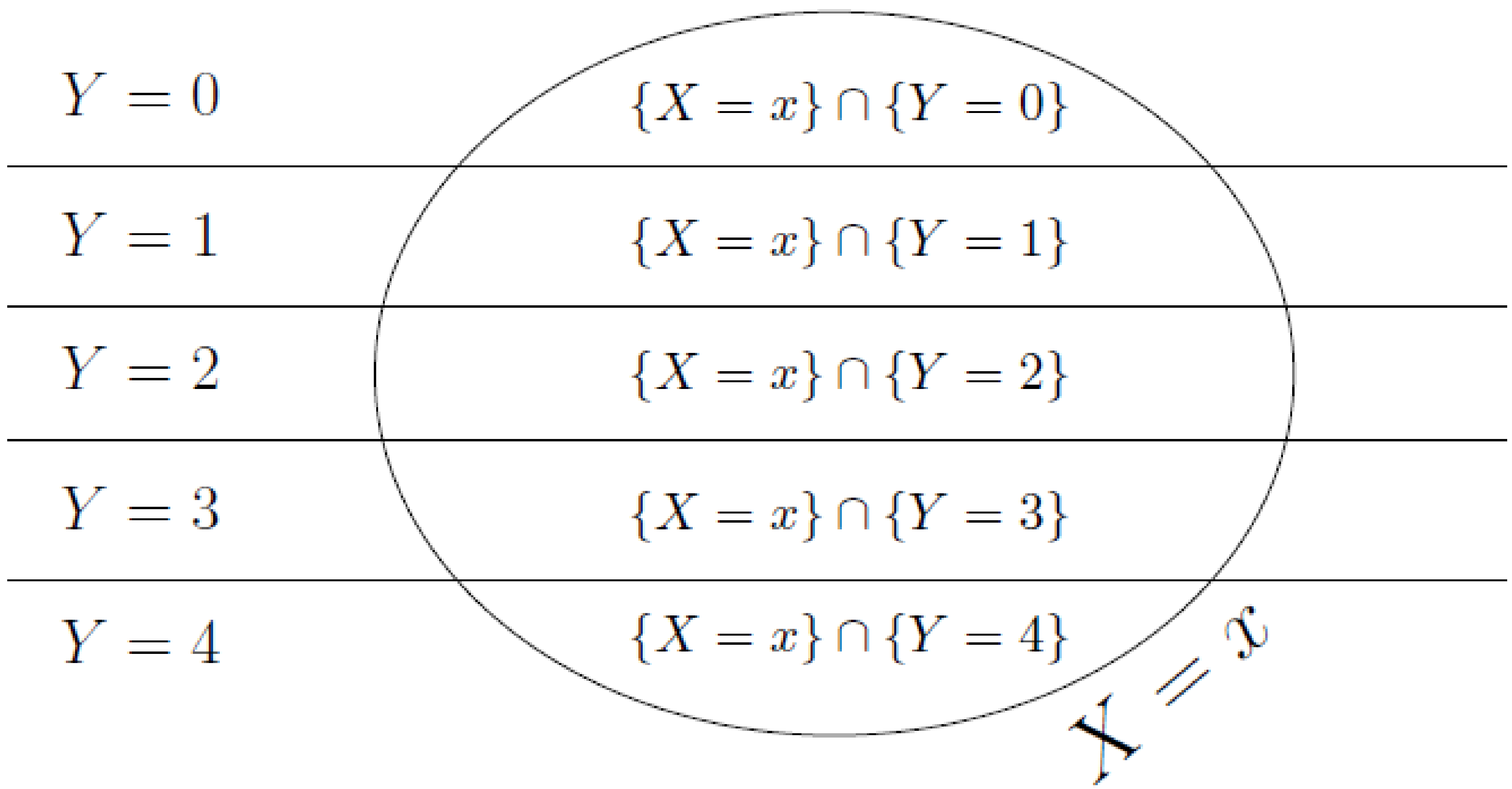
- Study of several random variables simultaneously
- If X and Y are two random variables, (X,Y) is a random vector
- The distribution of (X,Y) is called the *joint distribution* of X and Y.
- Given the joint distribution, the individual distribution of X and Y are called *marginal distributions*.

Joint p.m.f

- $f_{X,Y}(x,y) = P\big((X,Y) = (x,y)\big) = P(X = x \cap Y = y)$
- For all the possible and different pairs of (x,y) $\{(X,Y)=(x,y)\}$ are mutually exclusive and exhaustive events.
  - In other words:
    - $\sum_x \sum_y f_{X,Y}(x,y) = 1$
- The joint distribution has all the information about the random variables
  - Marginal (i.e., individual) probabilities can be computed

Addition Rule

- $f_X(x) = P(X = x) = \sum_y f_{X,Y}(x, y)$



Independent Random Variables

- In general, the joint pmf cannot be computed from marginal pmfs
- This is only possible if the random variables are independent
  - $f_{X,Y}(x, y) = f_X(x) f_Y(y)$
- Given the joint pmf, we can check independence of two random variables by checking whether the product of marginal pmfs is equal to the joint pmf for every pair of (x,y)

Example 3.6

- Are X and Y independent?

$P_{(X,Y)}(x,y)$		y = 0	1	2	3	$P_X(x)$
x	0	0.20	0.20	0.05	0.05	0.50
	1	0.20	0.10	0.10	0.10	0.50
$P_Y(y)$		0.40	0.30	0.15	0.15	1.00

Expected Value

- Expected value or the mean of a discrete r.v.

$$E(X) = \mu = \sum_{x} x f(x) = x_1 f(x_1) + x_2 f(x_2) \ldots\ldots$$

- Expected value of a continuous r.v.

$$E(X) = \mu = \int_{x} x f(x) dx$$

- $E(X)$ can be thought of as the center of gravity of the distribution of X

Expected Value

- Expected value of a function of X

$$E[g(x)] = \begin{cases} \sum_x g(x) f(x) & \text{if } X \text{ is discrete} \\ \int_x g(x) f(x) dx & \text{if } X \text{ is continuous} \end{cases}$$

- For example if $g(X) = X^2$

$$E[X^2] = \begin{cases} \sum_x X^2 f(x) & \text{if } X \text{ is discrete} \\ \int_x X^2 f(x) dx & \text{if } X \text{ is continuous} \end{cases}$$

Linearity of Expected Values

- For any two random variables X_1 and X_2 and any constants $c_1, c_2 \in \Re$

$$E(c_1 X_1 + c_2 X_2) = c_1 E(X_1) + c_2 E(X_2)$$

Expected value of a product

- In general $E(XY)$ is not equal to $E(X)E(Y)$

$$E(XY) = \int_y \int_x xyj(x, y)dxdy$$

- where $j(x,y)$ is the joint distribution which is NOT equal to $f(x)g(y)$ if X and Y are not independent.

Variance and Standard Deviation

- Variance of a r.v. is denoted by $Var(X)$ or σ^2

$$Var(X) = E[(X-\mu)^2] = E(X^2) - \mu^2$$

$$\sigma = \sqrt{Var(X)}$$

Example

- Experiment: two fair dice are tossed
- What is the expected value of the r.v. X?

  X = sum of two dice
- What is the variance?

Example

- Let X be a continuous r.v. with a p.d.f $f(X) = ½$ for $0<x<2$.
- What is the expected value of X?
- What is the variance of X?

Covariance

- A measure of strength of a relationship between two random variables
- E.g., X = height of a person p, Y = weight of the same person
- X and Y are paired random variables
- Is there a relationship between them?

$$Cov(X,Y) = E[(X-\mu_x)(Y-\mu_y)] = E(XY) - E(X)E(Y)$$

Correlation

- The correlation (or correlation coefficient) is simply the covariance standardized to the range of [-1,1]

$$Corr(X,Y) = \rho = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$