CEng 222
Statistical Methods for Computer Engineering

Week 5

Chapter 4

Continuous Distributions:

Gamma and Normal Distributions,

Central Limit Theorem

Gamma distribution

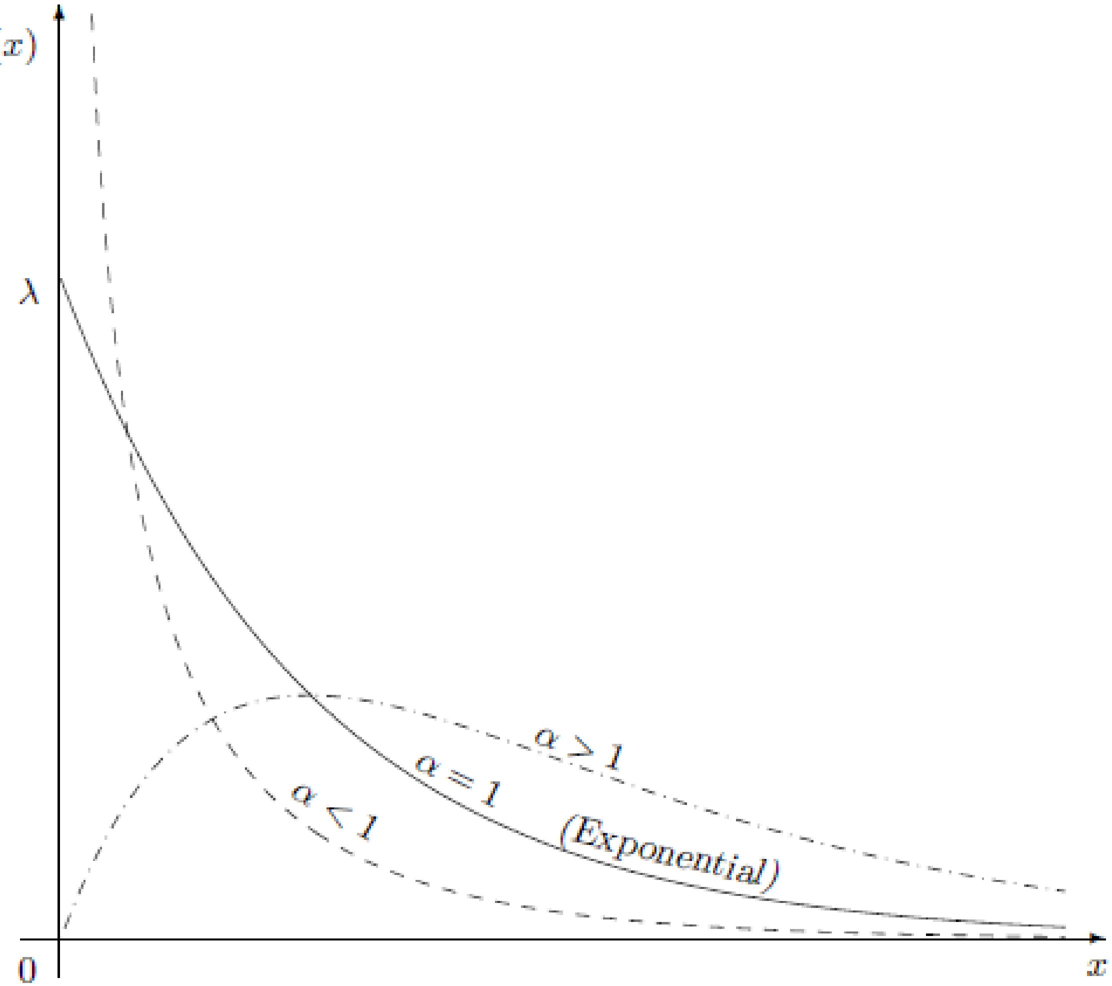
- X = the total time of observing α rare and independent events each with exponential waiting times (with parameter λ)
  - i.e., it is the sum of α exponential rvs

- Expectation and variance can be found using linearity of expectation.
  - $E(X) = \frac{\alpha}{\lambda}$, $Var(X) = \frac{\alpha}{\lambda^2}$

Gamma pdf

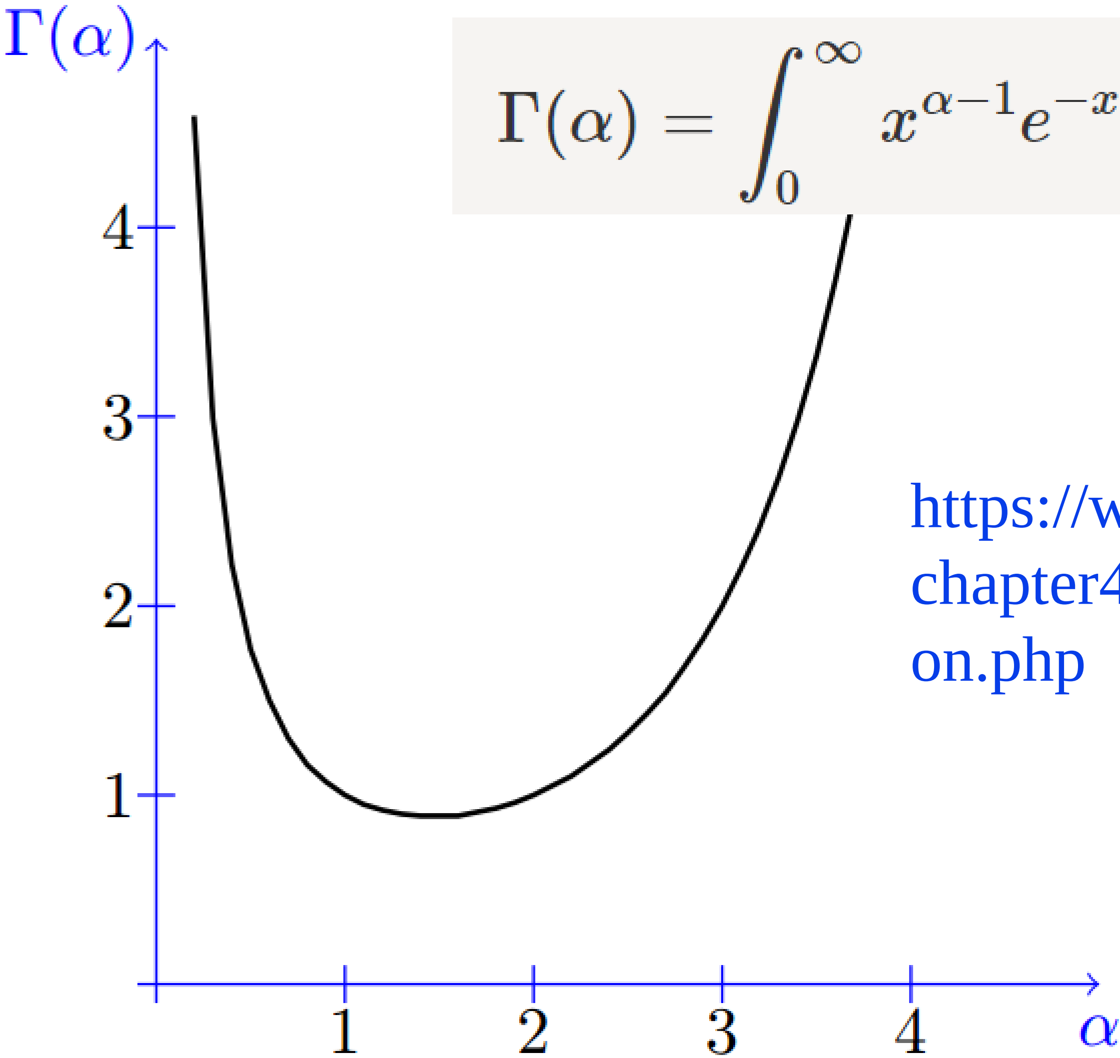
- $f_X(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda}, \quad x > 0$
- $\Gamma(\alpha) = (\alpha - 1)!$

α does not need to be an integer



The Gamma function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx, \qquad \text{for } \alpha > 0.$$



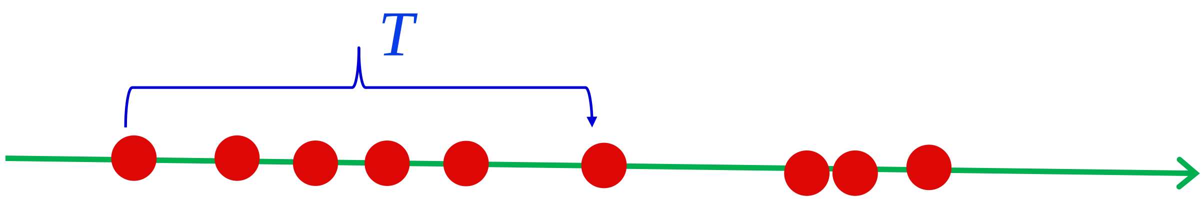
https://www.probabilitycourse.com/chapter4/4_2_4_Gamma_distribution.php

Gamma distribution

- Is widely used to model random variables other than waiting times (since α does not need to be an integer)
  - Amount of money spent
  - Amount of resources used (electricity, gas, etc.)

Gamma-Poisson formula

- Rare events



- T = time of the αth rare event = Gamma (α,λ)
  - The event $\{T>t\}$ means that fewer than α events occur in t time.
  - Let X be a Poisson rv with parameter λt
  - $\{T > t\} = \{X < \alpha\}$ hence $P(T > t) = P(X < \alpha)$
  - → $P(T \leq t) = P(X \geq \alpha)$
  - → we can use the Poisson table for computation of Gamma probabilities (Caution: T is continuous, X is discrete)

Example 4.9

- Lifetimes for computer chips have Gamma distribution with expectation μ=12 years and standard deviation σ=4 years. What is the probability that such a chip has a lifetime between 8 and 10 years?
- Step 1: what are the parameters of this Gamma rv?
  - $\frac{\alpha}{\lambda} = 12, \frac{\alpha}{\lambda^2} = 16 \rightarrow \lambda = 12/16 = 0.75, \alpha = 12*0.75 = 9$

Example 4.9 continued

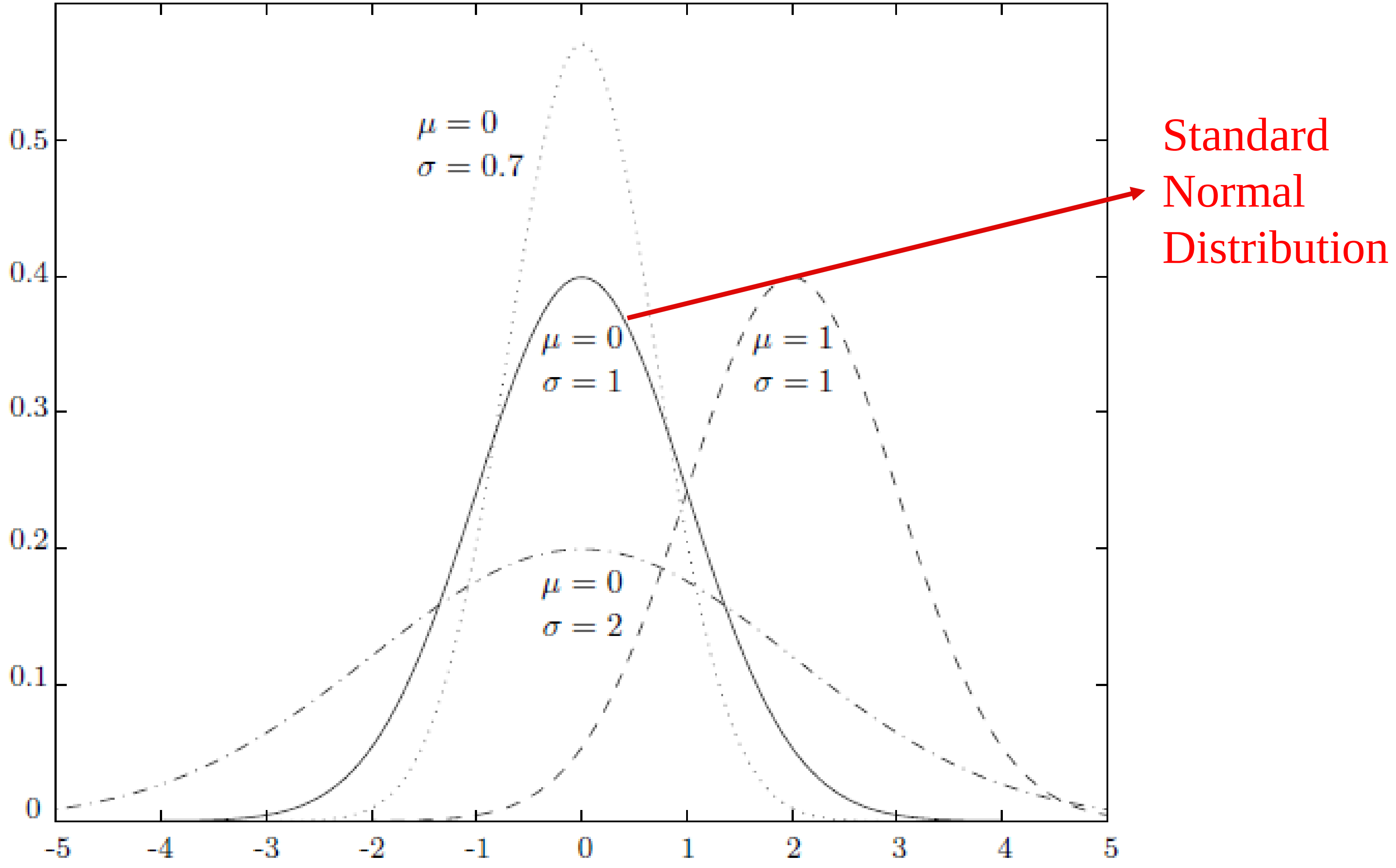
- Step 2: Compute the probability
  - $P(8<T<10) = F_T(10) - F_T(8)$
  - $F_T(10) = P(T \leq 10) = P(X_1 \geq 9)$ where $X_1 = Poisson(7.5)$
    - $P(X_1 \geq 9) = 1\text{-}F_{X_1}(8) = 0.338$
  - $F_T(8) = P(T \leq 8) = P(X_2 \geq 9)$ where $X_2 = Poisson(6)$
    - $P(X_2 \geq 9) = 1\text{-}F_{X_2}(8) = 0.153$
  - $P(8<T<10) = 0.338 - 0.153 = 0.185$

Normal (Gaussian) distribution

- A good model for physical variables like weight, height, temperature, etc.
- Sums and averages of arbitrarily distributed rvs are also normally distributed (Central Limit Theorem)
  - Thus, very popular for modelling errors
- Normal pdf:
  - $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\}$, $-\infty < x < +\infty$

Normal distribution

- The mean and the std. dev. are also called *location* and *scale* parameters.



Standard Normal Distribution

- Any non-standard Normal rv X with Normal(μ, σ) can be standardized as follows:
  - Z = Normal(0,1) = $\frac{X-\mu}{\sigma}$
  - and vice versa: $X = \mu + \sigma Z$
  - → we only need the Standard Normal Distribution table
- Example 4.11 – computing non-standard probabilities using the standard normal table
- Example 4.12 – solving inverse problems

Central Limit Theorem

- Let $X_1,\ldots,X_n$ be random variables from any distribution with $\mu=\mathbf{E}(X_i)$ and $\sigma^2=\mathbf{Var}(X_i)$ (n rvs from the same distribution)

As $n \rightarrow \infty$,

$$\frac{(X_1+\cdots+X_n)-n\mu}{\sigma\sqrt{n}} \rightarrow \text{Normal}(0,1)$$

$$\Rightarrow \text{P}\left(\frac{(X_1+\cdots+X_n)-n\mu}{\sigma\sqrt{n}} \le x\right) \to F_{\text{Normal}(0,1)}(x)$$

Examples:

Binomial(n,p) ≈ Normal(μ,σ) for large n

Gamma(α,λ) ≈ Normal(μ,σ) for large α

Central Limit Theorem

- Example 4.13
- Example 4.14

Normal Approximation to Binomial

- Binomial(n,p) ≈ Normal($\mu = np, \sigma = \sqrt{np(1-p)}$)
- We need continuity correction
  - P(X=x) = 0 for a continuous variable X
  - If we want to find $f_B(b)$ for a Binomial variable B
    - $f_B(b) = P(B = b) = P(b - 0.5 < B < b + 05)$
    - We expand the interval for the discrete variable 0.5 units in each direction and use the Normal approximation to compute the probability of an interval, not the probability of a point.
- Example 4.15