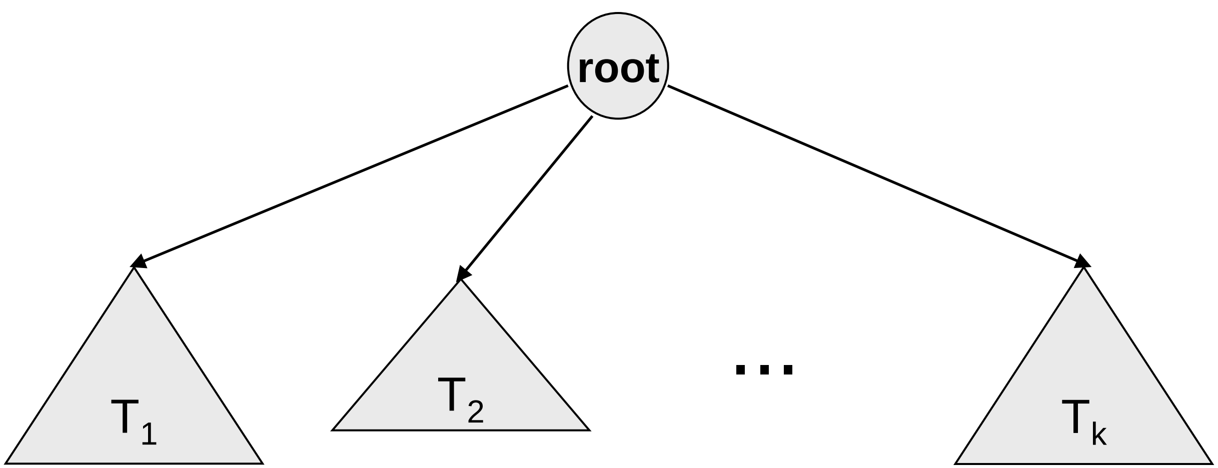
Trees

Outline

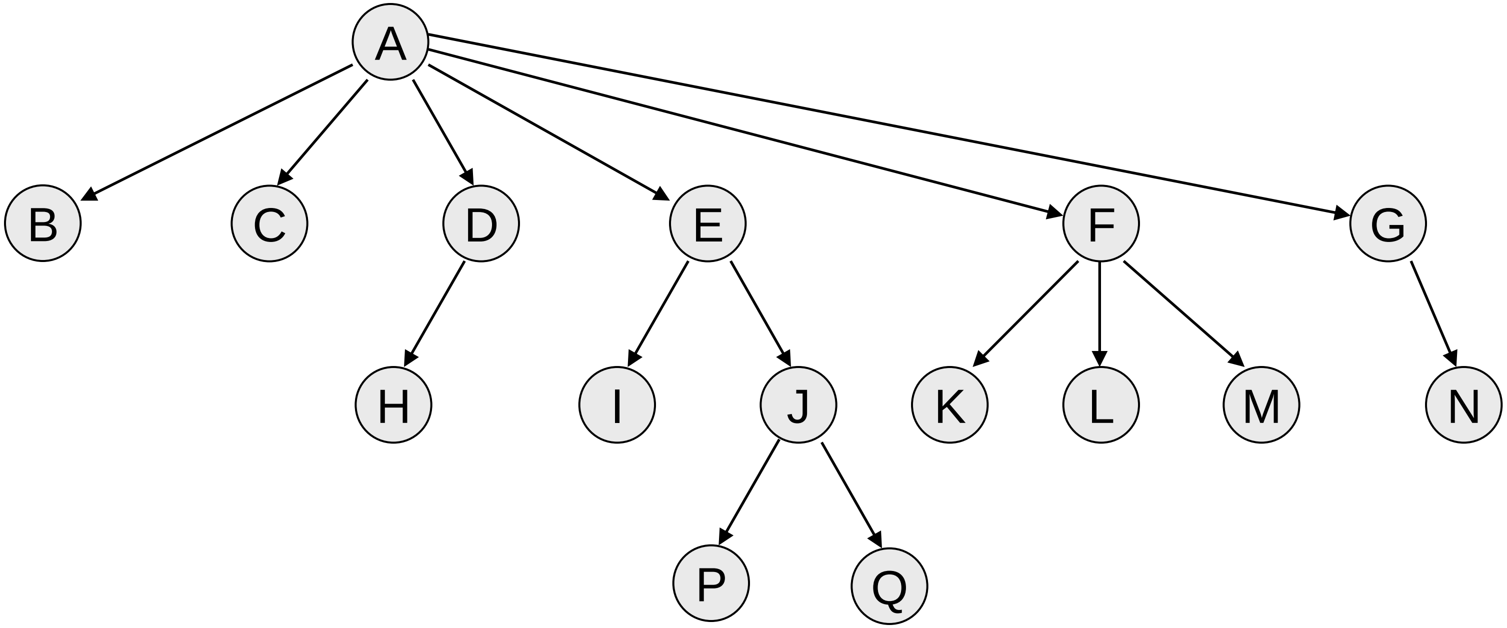
- Preliminaries
  - What is Tree?
  - Implementation of Trees using C++
  - Tree traversals and applications
- Binary Trees
- Binary Search Trees
  - Structure and operations
  - Analysis

What is a Tree?

- A tree is a collection of nodes with the following properties:
  - The collection can be empty.
  - Otherwise, a tree consists of a distinguished node r, called *root*, and zero or more nonempty sub-trees $T_1, T_2, \ldots, T_k$, each of whose roots are connected by a *directed edge* from r.
- The root of each sub-tree is said to be *child* of r, and r is the *parent* of each sub-tree root.
- If a tree is a collection of N nodes, then it has N-1 edges.



Preliminaries



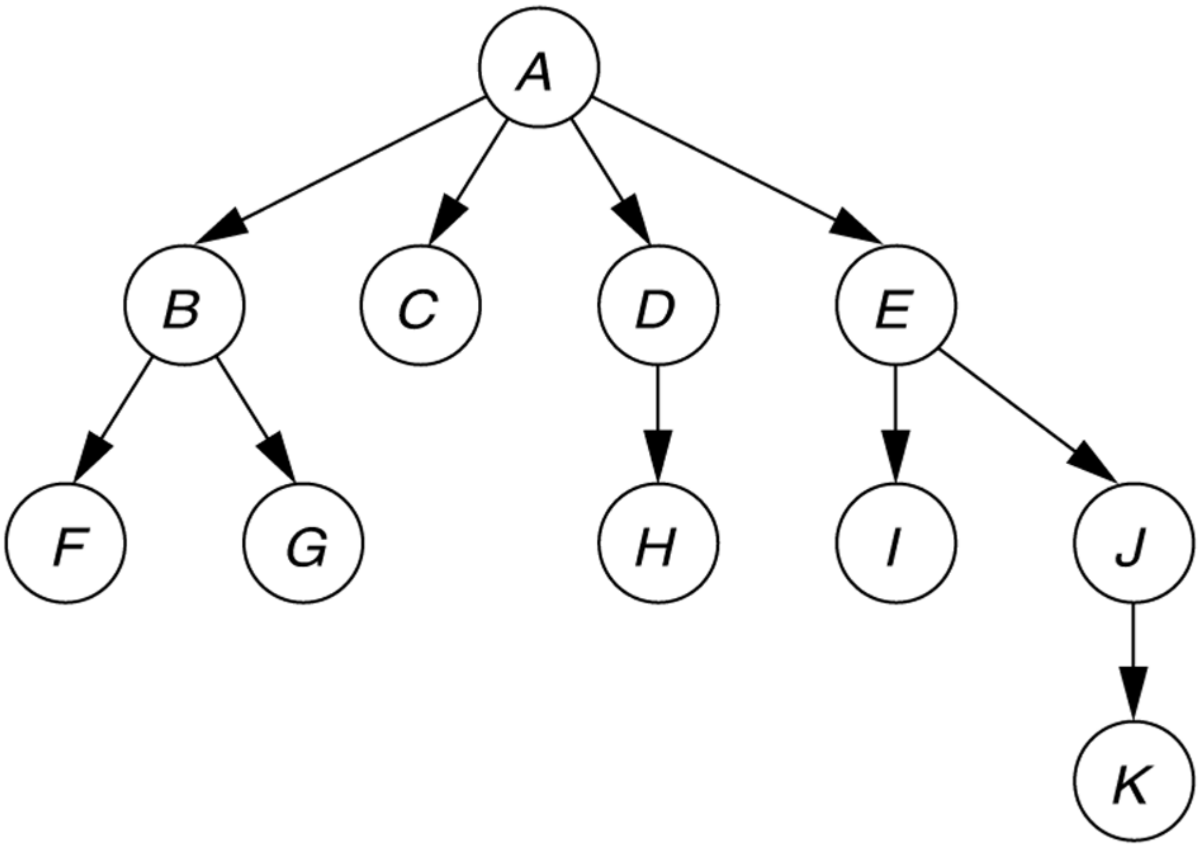
– Node *A* has 6 *children*: B, C, D, E, F, G.
– B, C, H, I, P, Q, K, L, M, N are *leaves* in the tree above.
– K, L, M are *siblings* since F is parent of all of them.

Preliminaries (continued)

- A ***path*** from node n_1 to n_k is defined as a sequence of nodes n_1, n_2, …, n_k such that n_i is parent of n_{i+1} $(1 \leq i < k)$
  - The ***length*** of a path is the number of edges on that path.
  - There is a path of length zero from every node to itself.
  - There is exactly one path from the root to each node.
- The ***depth*** of node n_i is the length of the path from *root* to node n_i
- The ***height*** of node n_i is the length of longest path from node n_i to a *leaf*.
- If there is a path from n_1 to n_2, then n_1 is ***ancestor*** of n_2, and n_2 is ***descendent*** of n_1.
  - If $n_1 \neq n_2$ then n_1 is *proper ancestor* of n_2, and n_2 is *proper descendent* of n_1.

Figure 1

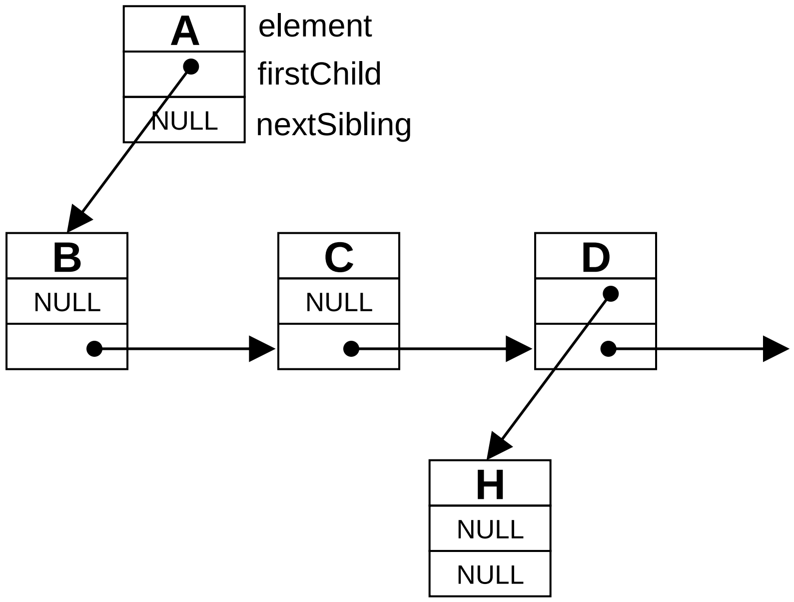
A tree, with height and depth information



Node	Height	Depth
A	3	0
B	1	1
C	0	1
D	1	1
E	2	1
F	0	2
G	0	2
H	0	2
I	0	2
J	1	2
K	0	3

Implementation of Trees

```
struct TreeNode {
    Object       element;
    struct TreeNode *firstChild;
    struct TreeNode *nextSibling;
};
```



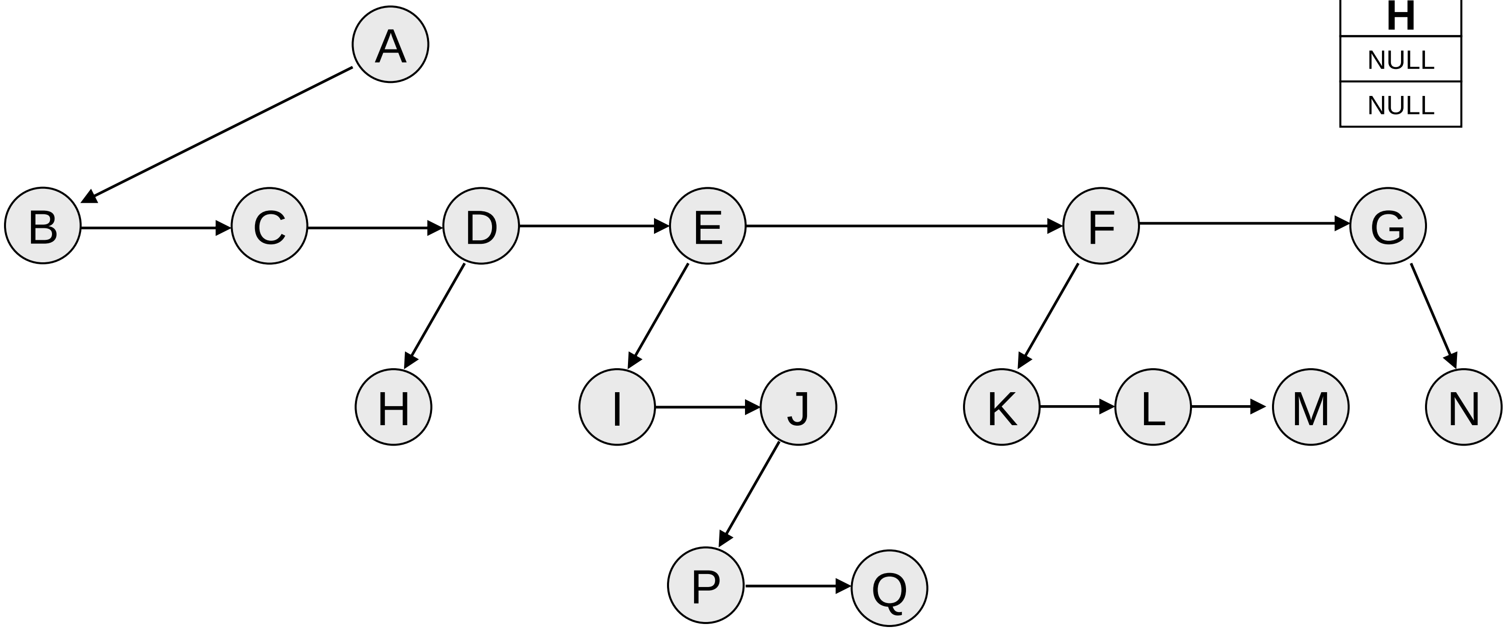
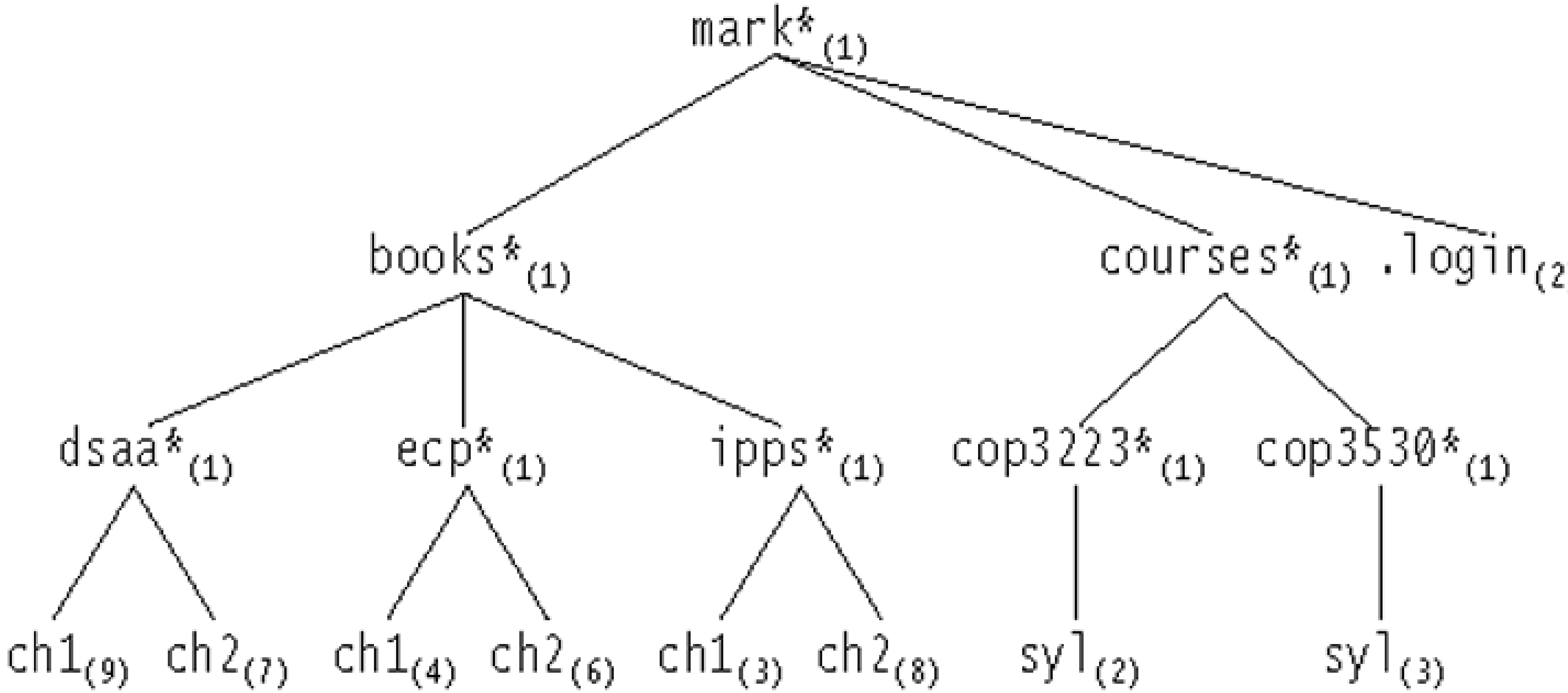


Figure 2: The Unix directory with file sizes



Listing a directory

```
// Algorithm (not a complete C code)
listAll ( struct TreeNode *t, int depth)
{
     printName ( t, depth );
     if (isDirectory())
        for each file c in  this directory (for each child)
            listAll(c, depth+1 );
}
```

- printName() function prints the name of the object after "depth" number of tabs -indentation. In this way, the output is nicely formatted on the screen.
- The order of visiting the nodes in a tree is important while traversing a tree.
  - Here, the nodes are visited according to *preorder* traversal strategy.

Figure 3: The directory listing for the tree shown in Figure 2

```
mark
        books
                dsaa
                        ch1
                        ch2
                ecp
                        ch1
                        ch2
                ipps
                        ch1
                        ch2
        courses
                cop3223
                        syl
                cop3530
                        syl
        .login
```

Size of a directory

```
int FileSystem::size () const
{
    int totalSize   = sizeOfThisFile();

    if (isDirectory())
       for each file c in  this directory (for each child)
            totalSize += c.size();
    return totalSize;
}
```

- The nodes are visited using *postorder* strategy.
- The work at a node is done after processing each child of that node.

Figure 18.9

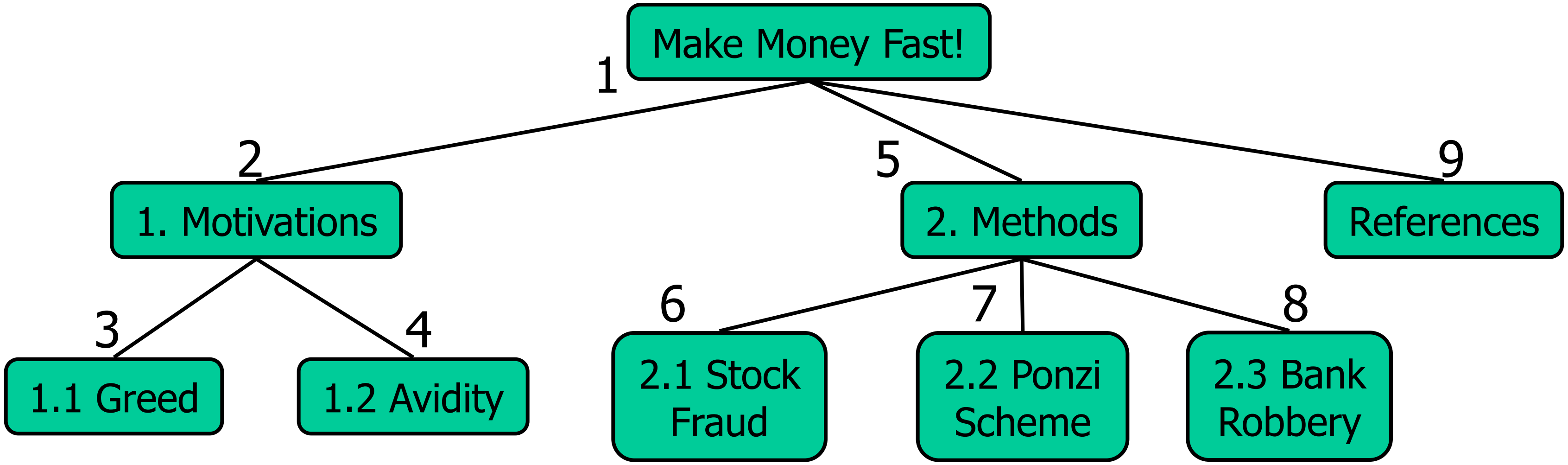
A trace of the size method

```
                        ch1         9
                        ch2         7
                dsaa               17
                        ch1         4
                        ch2         6
                ecp                11
                        ch1         3
                        ch2         8
                ipps               12
        books                      41
                        syl         2
                cop3223             3
                        syl         3
                cop3530             4
        courses                     8
        .login                      2
mark                               52
```

Preorder Traversal

- A traversal visits the nodes of a tree in a systematic manner
- In a preorder traversal, a node is visited before its descendants
- Application: print a structured document

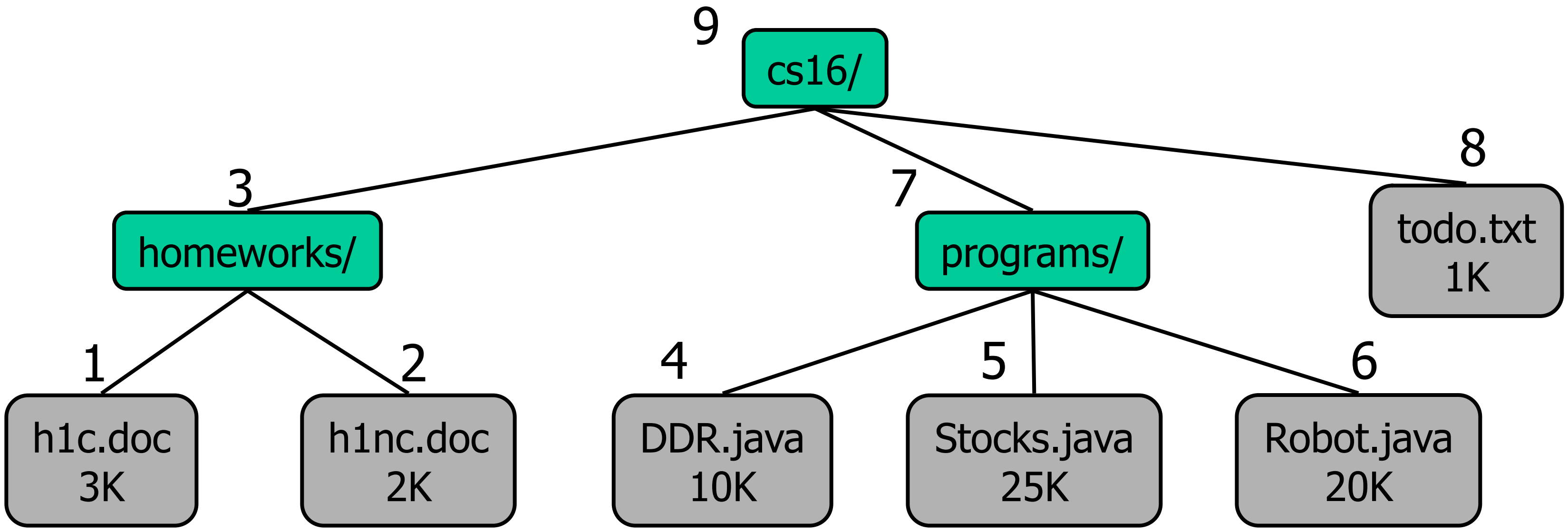
Algorithm ***preOrder***(*v*)
    visit(*v*)
    for each child *w* of *v*
        preorder (*w*)



Postorder Traversal

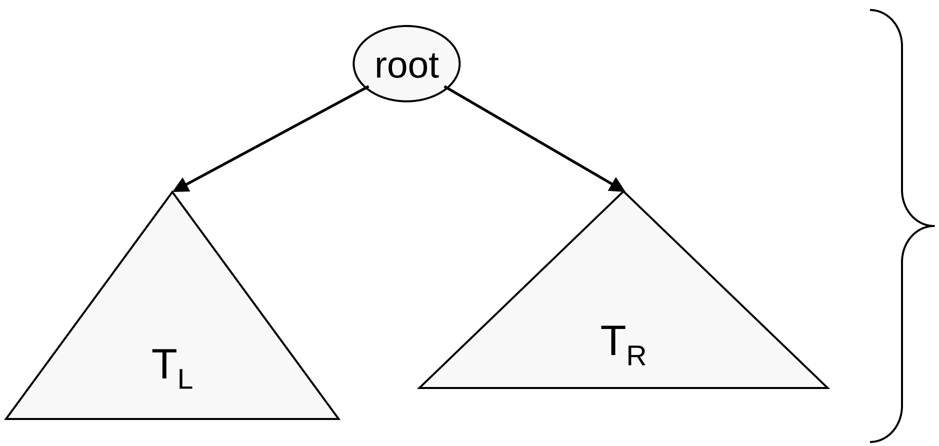
- In a postorder traversal, a node is visited after its descendants
- Application: compute space used by files in a directory and its subdirectories

Algorithm ***postOrder***(*v*)
    for each child ***w*** of ***v***
        postOrder (***w***)
    visit(***v***)



Binary Trees

- A *binary tree* is a tree in which no node can have more than two children
- The depth can be as large as *N-1* in the worst case.



A binary tree consisting of a root and two subtrees T_L and T_R, both of which could possibly be empty.

Binary Tree Terminology

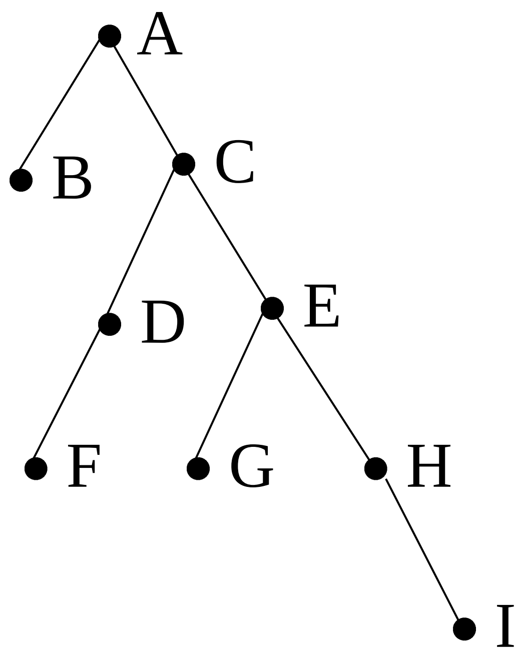
Left Child – The left child of node n is a node directly below and to the left of node n in a binary tree.

Right Child – The right child of node n is a node directly below and to the right of node n in a binary tree.

Left Subtree – In a binary tree, the left subtree of node n is the left child (if any) of node n plus its descendants.

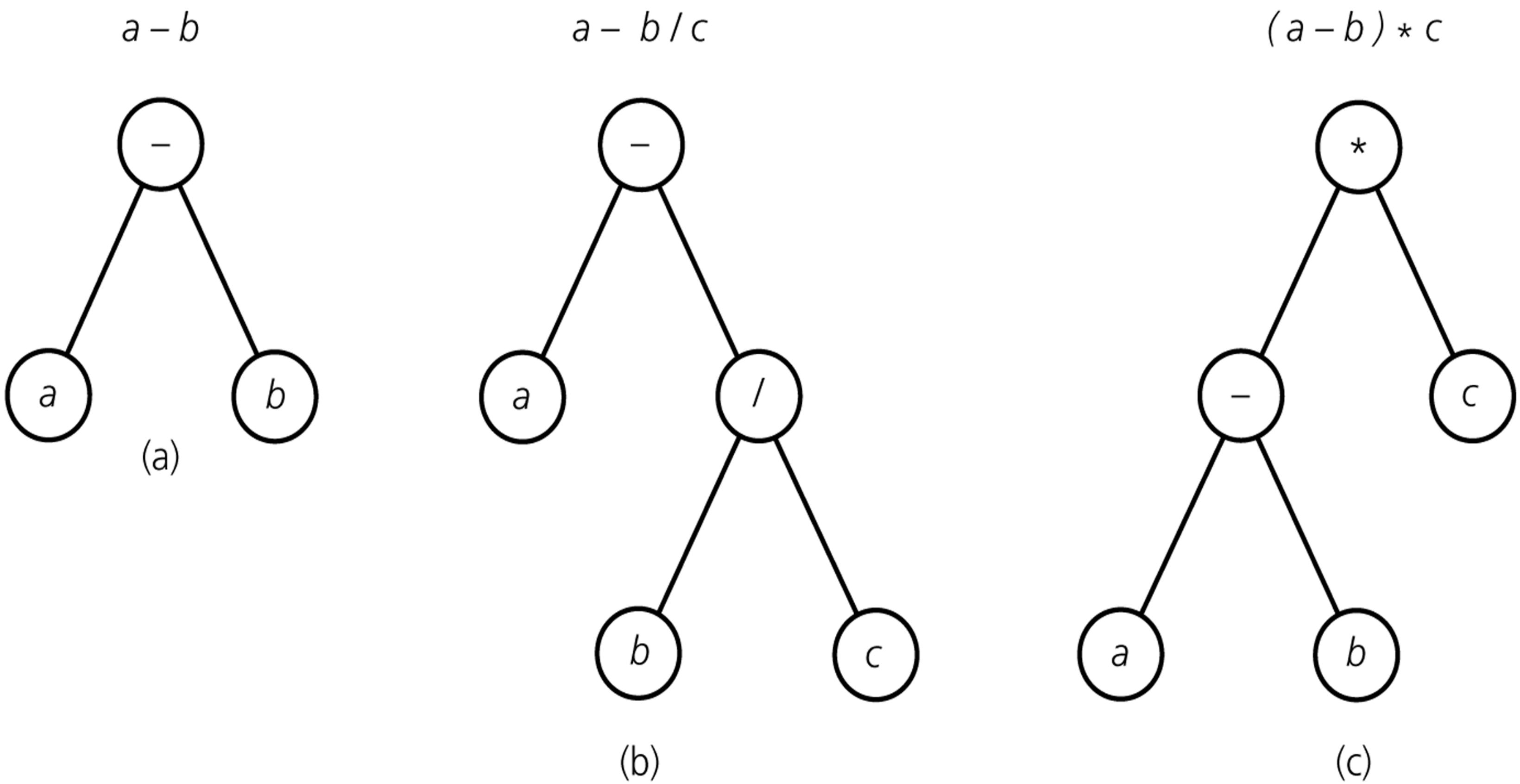
Right Subtree – In a binary tree, the right subtree of node n is the right child (if any) of node n plus its descendants.

Binary Tree -- Example



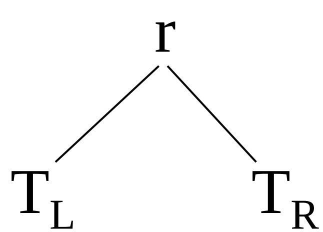
- A is the root.
- B is the left child of A, and C is the right child of A.
- D doesn't have a right child.
- H doesn't have a left child.
- B, F, G and I are leaves.

Binary Tree – Representing Algebraic Expressions



Height of Binary Tree

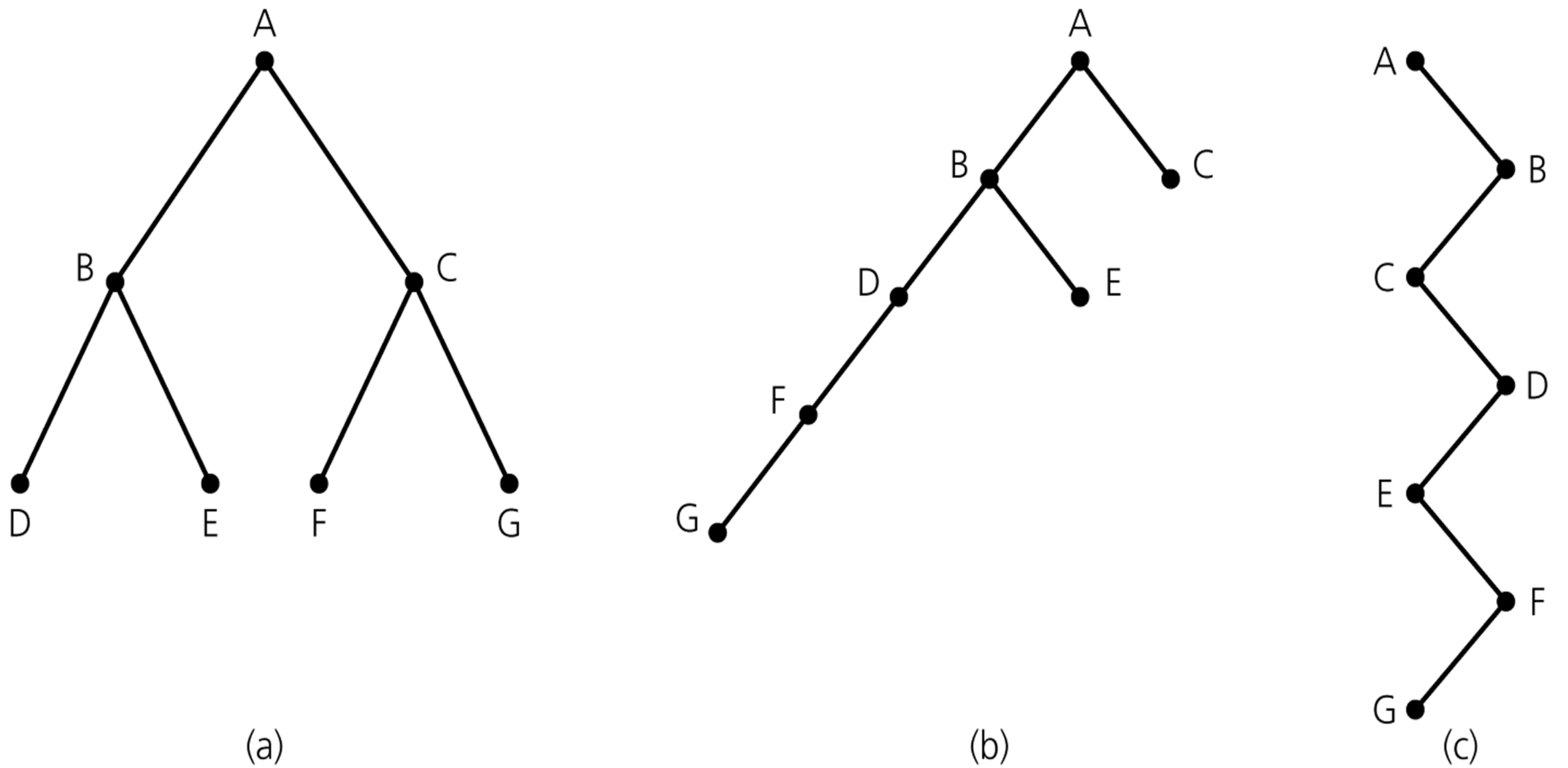
- The height of a binary tree T can be defined recursively as:
  - If T is empty, its height is -1.
  - If T is non-empty tree, then since T is of the form



the height of T is 1 greater than the height of its root's taller subtree; i.e.

$$height(T) = 1 + max\{height(T_L), height(T_R)\}$$

Height of Binary Tree (cont.)

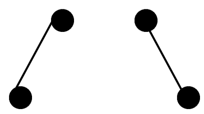


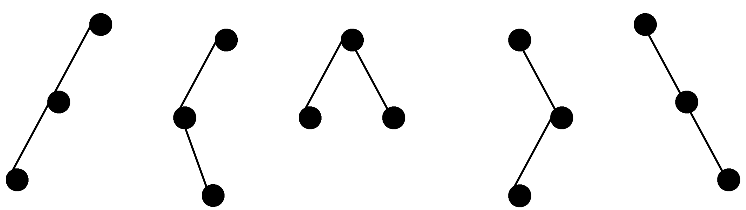
Binary trees with the same nodes but different heights

Number of Binary trees with Same # of Nodes

n=0 ➔ empty tree

n=1 ➔  (1 tree)

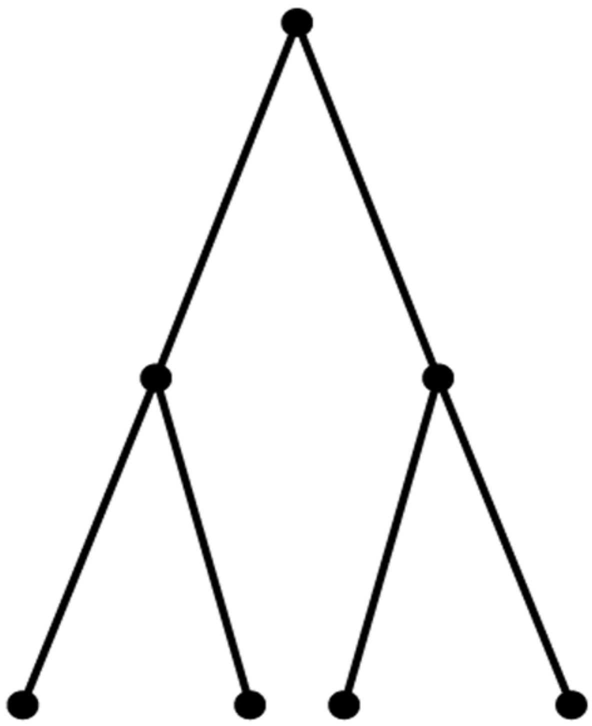
n=2 ➔  (2 trees)

n=3 ➔  (5 trees)

Full Binary Tree

- In a ***full binary tree*** of height h, all nodes that are at a level less than h have two children each.
- Each node in a full binary tree has left and right subtrees of the same height.
- Among binary trees of height h, a full binary tree has as many leaves as possible, and they all are at level h.
- A full binary has no missing nodes.
- Recursive definition of full binary tree:
  – If T is empty, T is a full binary tree of height -1.
  – If T is not empty and has height h>0, T is a full binary tree if its root’s subtrees are both full binary trees of height h-1.

Full Binary Tree – Example



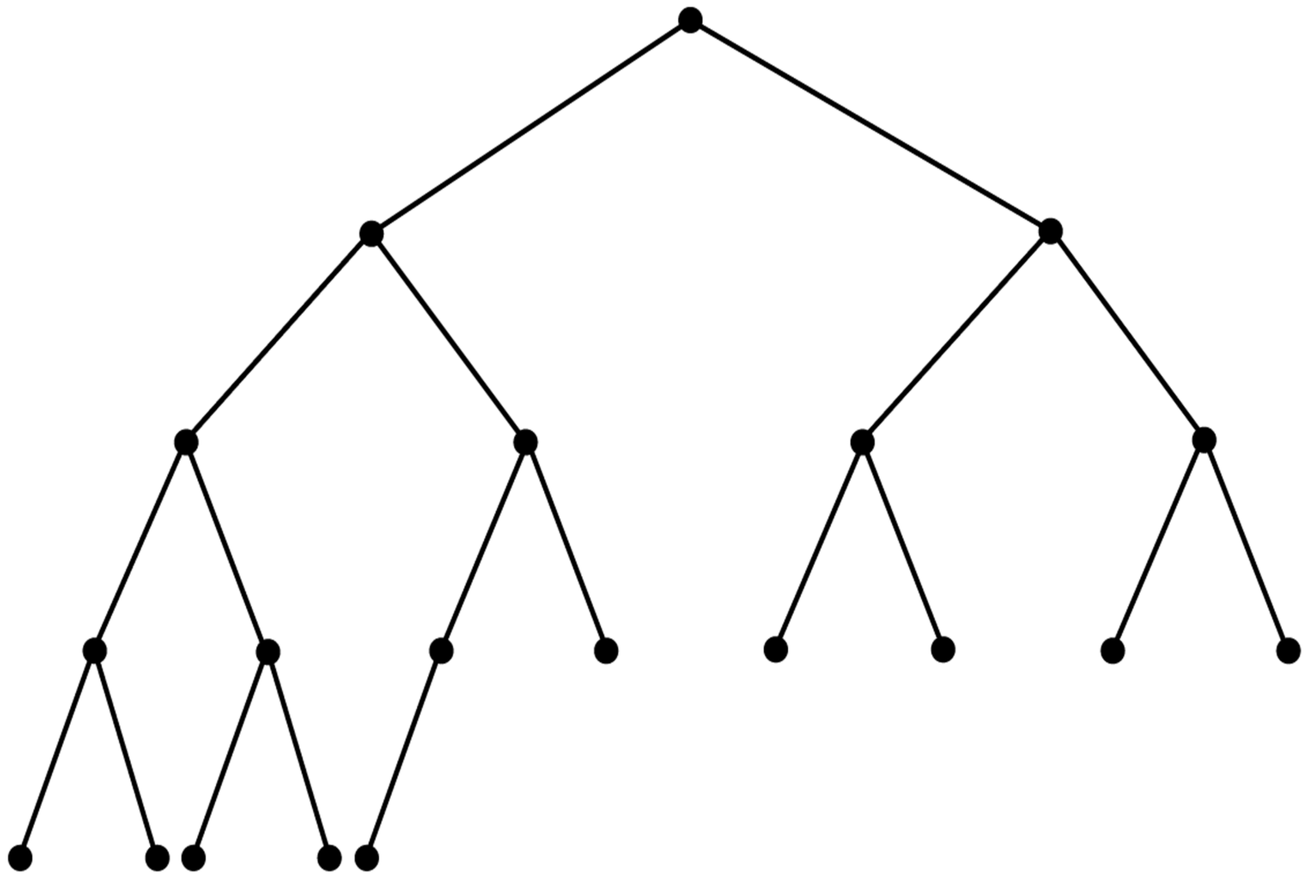
A full binary tree of height 2

Complete Binary Tree

- A ***complete binary tree*** of height h is a binary tree that is full down to level h-1, with level h filled in from left to right.
- A binary tree T of height h is complete if
  1. All nodes at level h-2 and above have two children each, and
  2. When a node at level h-1 has children, all nodes to its left at the same level have two children each, and
  3. When a node at level h-1 has one child, it is a left child.

– A full binary tree is a complete binary tree.

Complete Binary Tree – Example



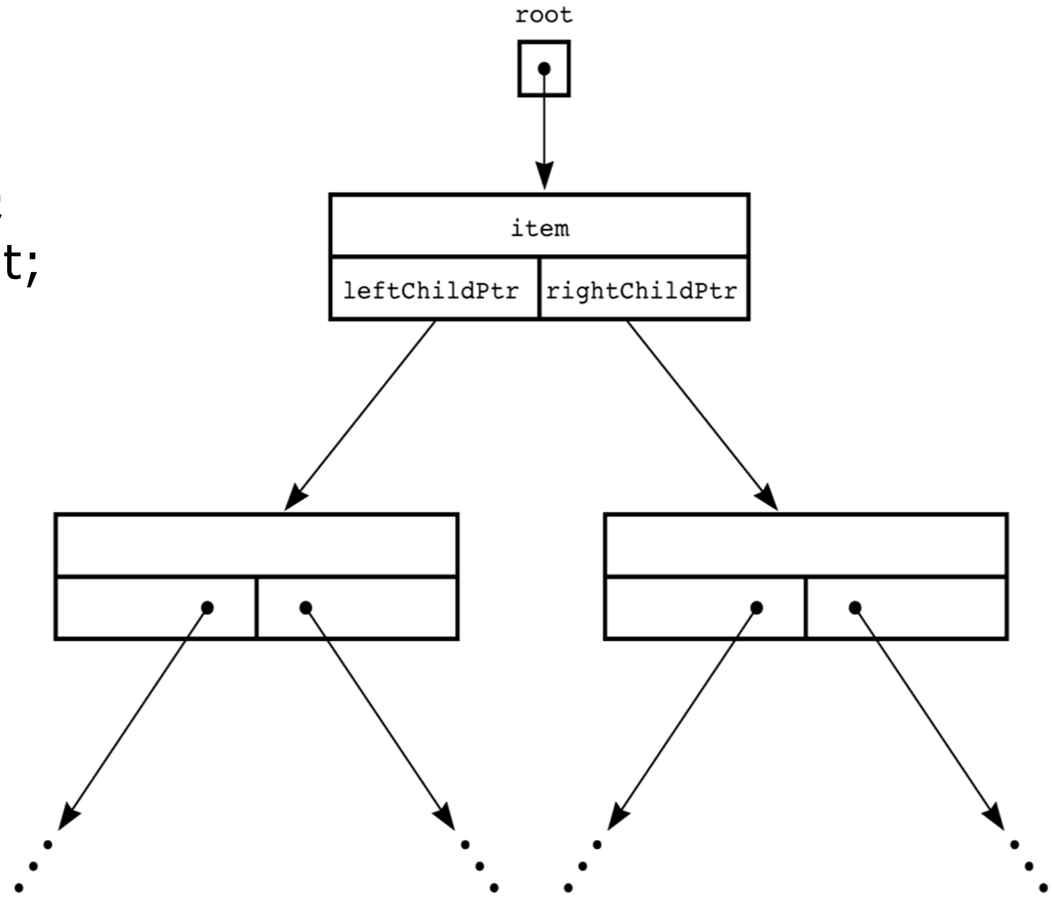
Balanced Binary Tree

- A binary tree is ***height balanced*** (or ***balanced***), if the height of any node's right subtree differs from the height of the node's left subtree by no more than 1.
- A complete binary tree is a balanced tree.
- Other height balanced trees:
  - AVL trees
  - Red-Black trees
  - B-trees

    ....

A Pointer-Based Implementation of Binary Trees

```
struct BinaryNode {
   Object          element;
   struct BinaryNode *left;
   struct BinaryNode *right;
};
```



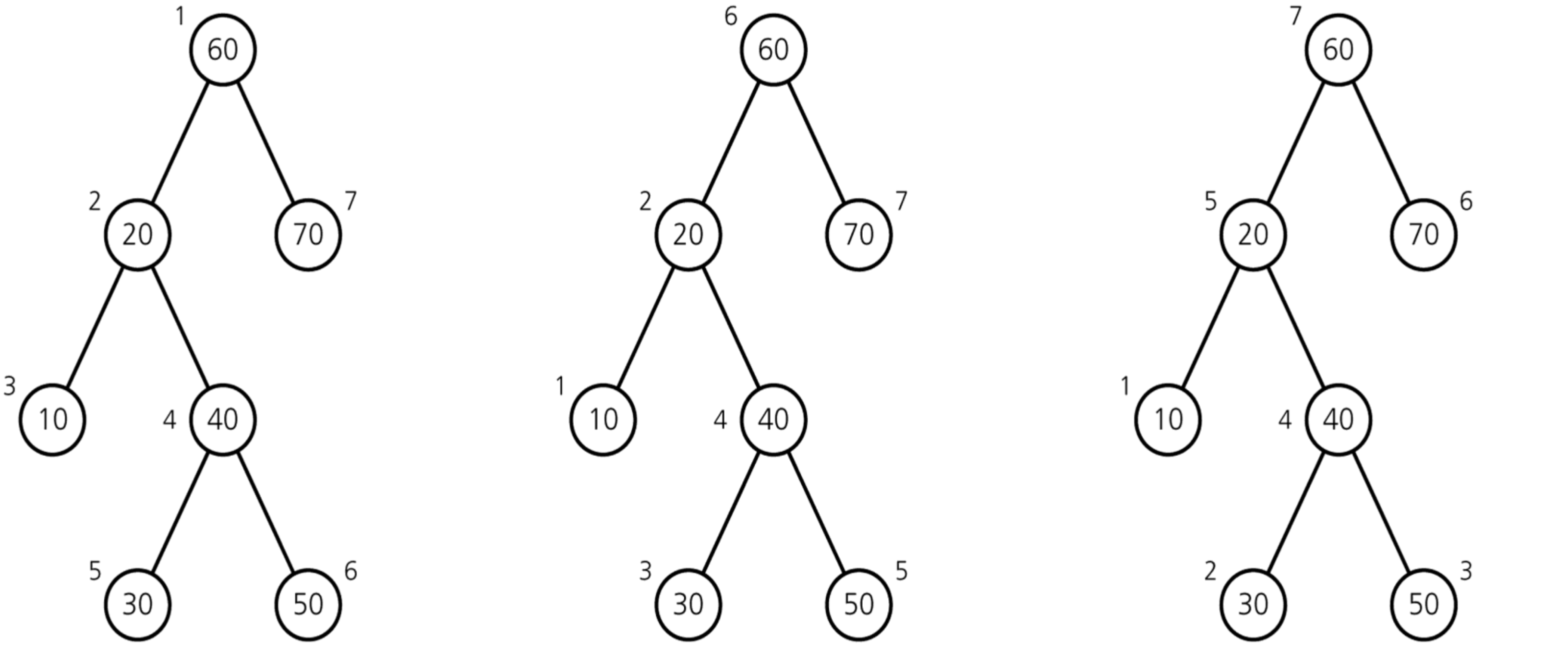
Binary Tree Traversals

- **Preorder Traversal**
  - the node is visited before its left and right subtrees,

- **Postorder Traversal**
  - the node is visited after both subtrees.

- **Inorder Traversal**
  - the node is visited between the subtrees,
  - Visit left subtree, visit the node, and visit the right subtree.

Binary Tree Traversals



(a) Preorder: 60, 20, 10, 40, 30, 50, 70

(b) Inorder: 10, 20, 30, 40, 50, 60, 70

(c) Postorder: 10, 30, 50, 40, 20, 70, 60

(Numbers beside nodes indicate traversal order.)

Preorder

```
void preorder(struct tree_node * p)
{ if (p !=NULL) {
     printf("%d\n", p->data);
     preorder(p->left_child);
     preorder(p->right_child);
}
}
```

Inorder

```
void inorder(struct tree_node *p)
{ if (p !=NULL) {
       inorder(p->left_child);
      printf("%d\n", p->data);
       inorder(p->right_child);
}
}
```

Postorder

```
void postorder(struct tree_node *p)
{ if (p !=NULL) {
      postorder(p->left_child);
      postorder(p->right_child);
      printf(“%d\n”, p->data);

}
}
```

Finding the maximum value in a binary tree

```
int FindMax(struct tree_node *p)
{
     int root_val, left, right, max;
     max = -1; // Assuming all values are positive integers

     if (p!=NULL) {
       root_val = p -> data;
       left = FindMax(p ->left_child);
       right = FindMax(p->right_child);

       // Find the largest of the three values.
       if (left > right)
            max = left;
       else
           max = right;
       if (root_val > max)
           max = root_val;
     }
     return max;
}
```

Adding up all values in a Binary Tree

```
int add(struct tree_node *p)
{

   if (p == NULL)
      return 0;
   else
      return (p->data + add(p->left_child)+
            add(p->right_child));
}
```

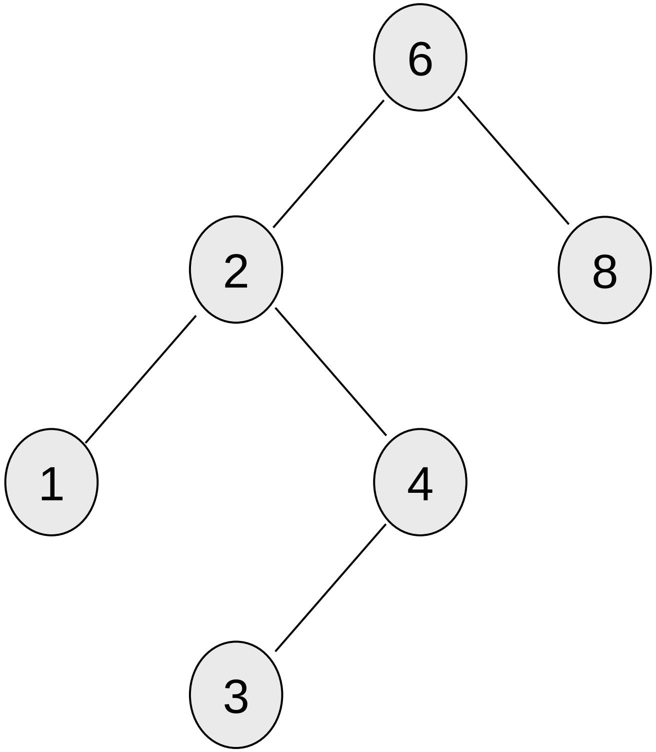
Exercises

1. Write a function that will count the leaves of a binary tree.
2. Write a function that will find the height of a binary tree.
3. Write a function that will interchange all left and right subtrees in a binary tree.

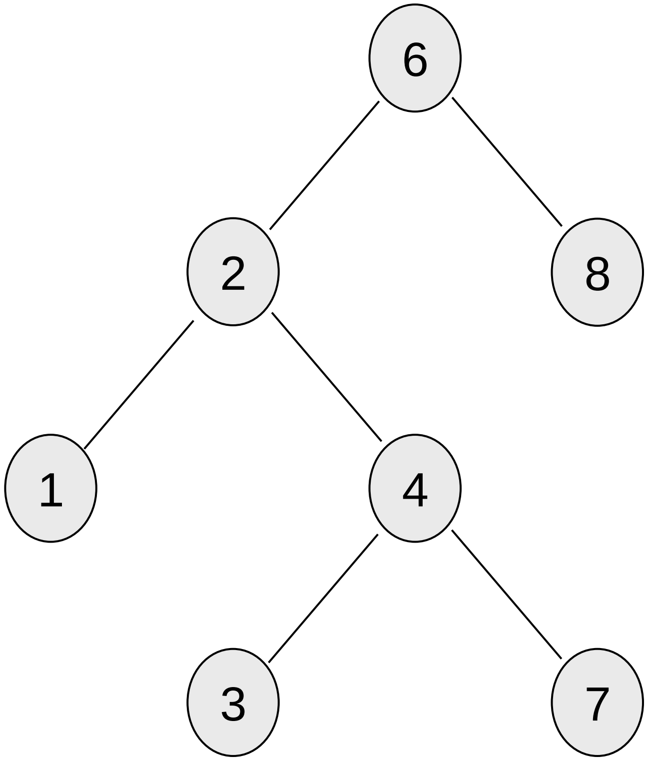
Binary Search Trees

- An important application of binary trees is their use in searching.
- *Binary search tree* is a binary tree in which every node X contains a data value that satisfies the following:
  a) all data values in its left subtree are smaller than the data value in X
  b) the data value in X is smaller than all the values in its right subtree.
  c) the left and right subtrees are also binary search tees.

Example

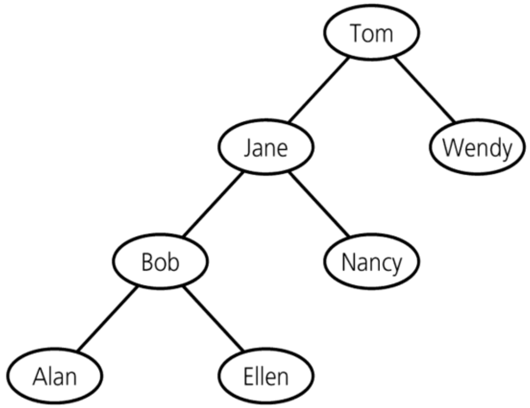


A *binary search tree*

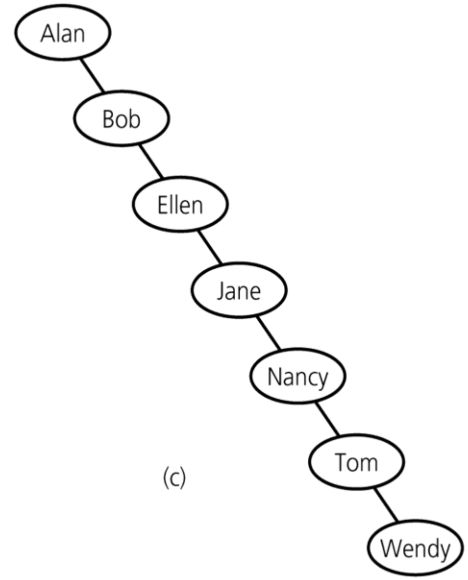


Not a *binary search tree,* but a *binary tree*

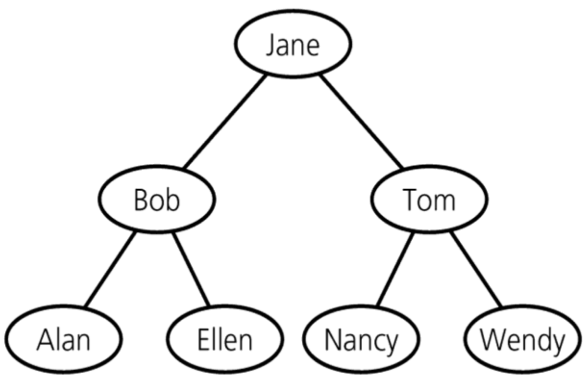
Binary Search Trees – containing same data

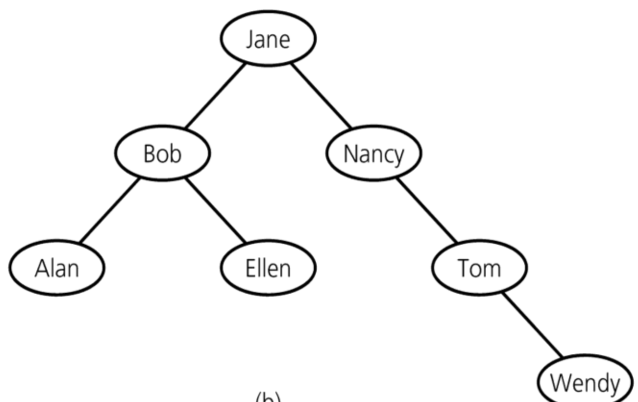


(a)



(c)





(b)

Operations on BSTs

- Most of the operations on binary trees are O(log*N*).
  - This is the main motivation for using binary trees rather than using ordinary lists to store items.
- Most of the operations can be implemented using recursion.
  - we generally do not need to worry about running out of stack space, since the average depth of binary search trees is O(log*N*).

The BinaryNode class

```
template <class Comparable>
class BinaryNode
{
   Comparable element;   // this is the item stored in the node
   BinaryNode *left;
   BinaryNode *right;

   BinaryNode( const Comparable & theElement, BinaryNode *lt,
     BinaryNode *rt ) : element( theElement ), left( lt ),
      right( rt ) { }
};
```

find

```
/**
 * Method to find an item in a subtree.
 * x is item to search for.
 * t is the node that roots the tree.
 * Return node containing the matched item.
 */
template <class Comparable>
BinaryNode<Comparable> *
find( const Comparable & x, BinaryNode<Comparable> *t ) const
{
  if( t == NULL )
      return NULL;
  else if( x < t->element )
      return find( x, t->left );
  else if( t->element < x )
      return find( x, t->right );
  else
      return t;    // Match
}
```

findMin (recursive implementation)

```
/**
 * method to find the smallest item in a subtree t.
 * Return node containing the smallest item.
 */
template <class Comparable>
BinaryNode<Comparable> *
findMin( BinaryNode<Comparable> *t ) const
{
   if( t == NULL )
       return NULL;
   if( t->left == NULL )
       return t;
   return findMin( t->left );
}
```

findMax (nonrecursive implementation)

```
/**
 *method to find the largest item in a subtree t.
 *Return node containing the largest item.
 */
template <class Comparable>
BinaryNode<Comparable> *
findMax( BinaryNode<Comparable> *t ) const
{
  if( t != NULL )
    while( t->right != NULL )
       t = t->right;
  return t;
}
```

Insert operation

Algorithm for inserting X into tree T:

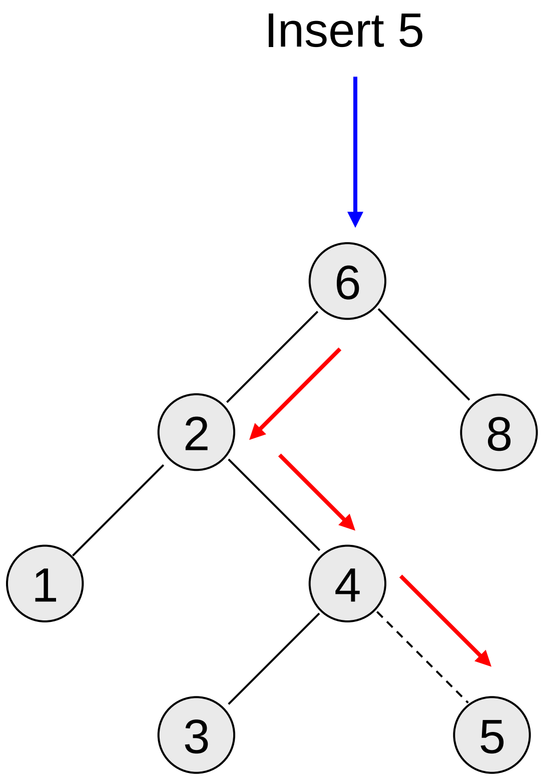
- Proceed down the tree as you would with a find operation.
- if X is found

    do nothing, (or "update" something)

  else

    insert X at the last spot on the path traversed.

Example



- What about duplicates?

Insertion into a BST

```
/* method to insert into a subtree.
 * x is the item to insert.
 * t is the node that roots the tree.
 * Set the new root.
 */
template <class Comparable>
void insert( const Comparable & x,
             BinaryNode<Comparable> * & t ) const
{
   if( t == NULL )
      t = new BinaryNode<Comparable>( x, NULL, NULL );
   else if( x < t->element )
      insert( x, t->left );
   else if( t->element < x )
      insert( x, t->right );
   else
      ;  // Duplicate; do nothing
}
```

Deletion operation

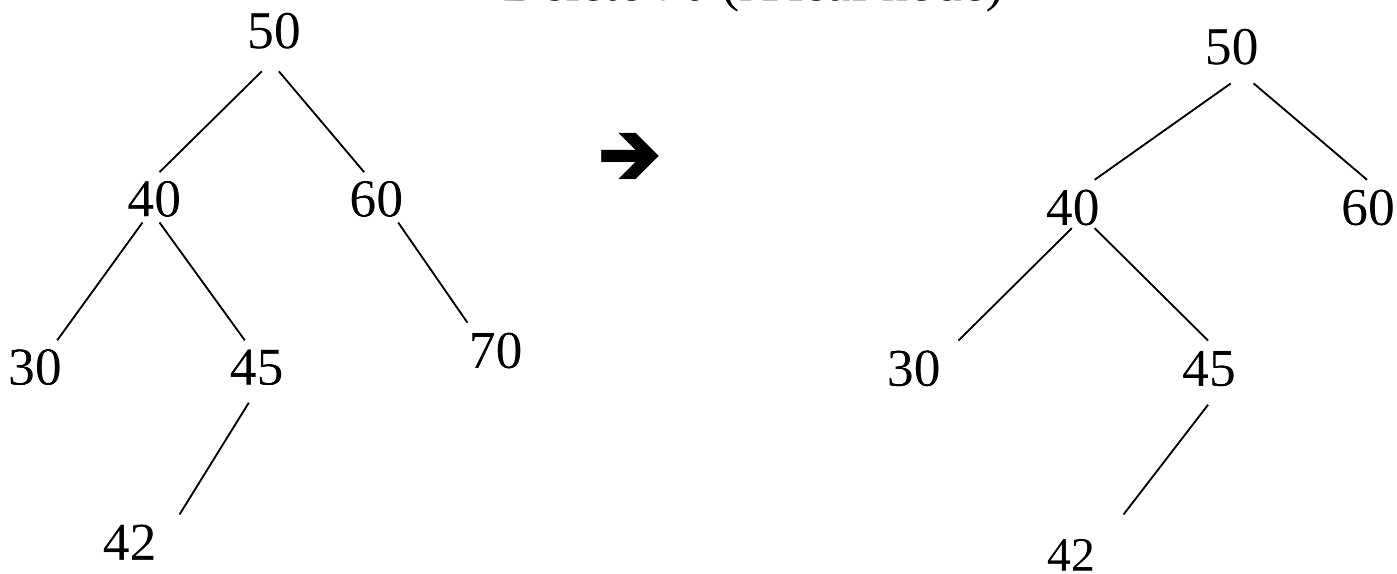
There are three cases to consider:

1. Deleting a leaf node
   - Replace the link to the deleted node by NULL.
2. Deleting a node with one child:
   - The node can be deleted after its parent adjusts a link to bypass the node.
3. Deleting a node with two children:
   - The deleted value must be replaced by an existing value that is either one of the following:
     - The largest value in the deleted node's left subtree
     - The smallest value in the deleted node's right subtree.

Deletion – Case1: A Leaf Node

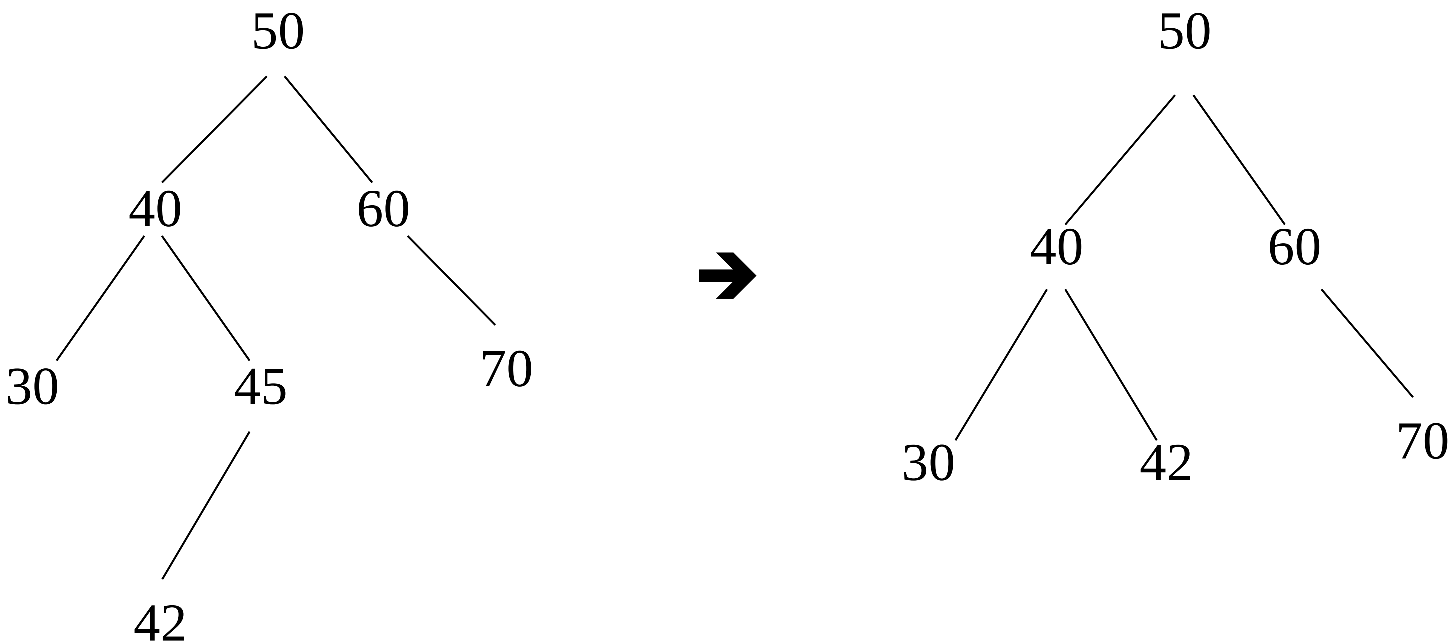
To remove the leaf containing the item, we have to set the pointer in its parent to NULL.

Delete 70 (A leaf node)



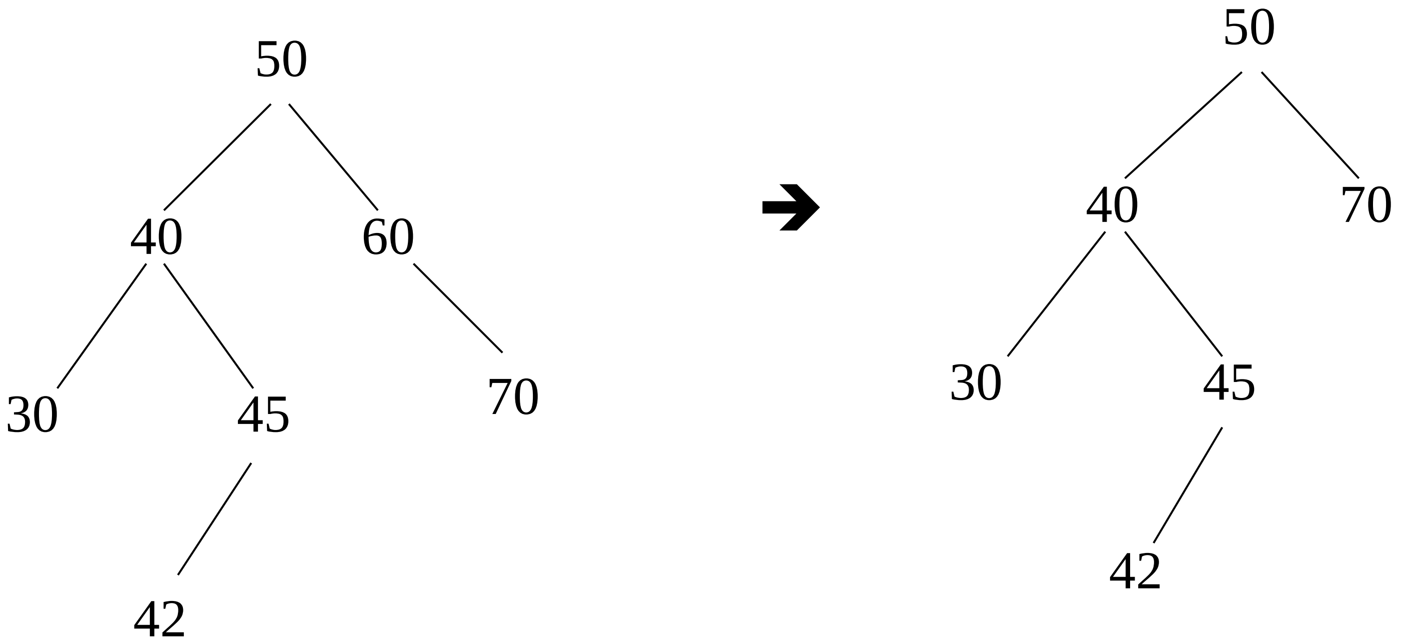
Deletion – Case2: A Node with only a left child

Delete 45 (A node with only a left child)



Deletion – Case2: A Node with only a right child

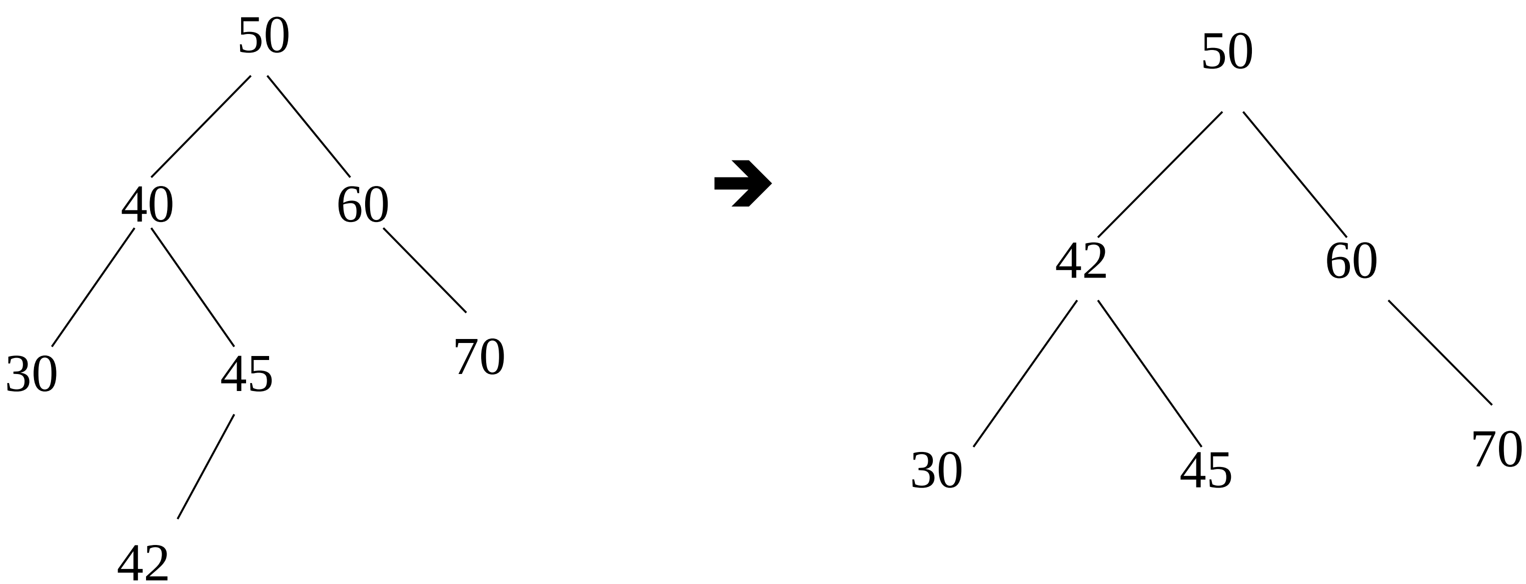
Delete 60 (A node with only a right child)



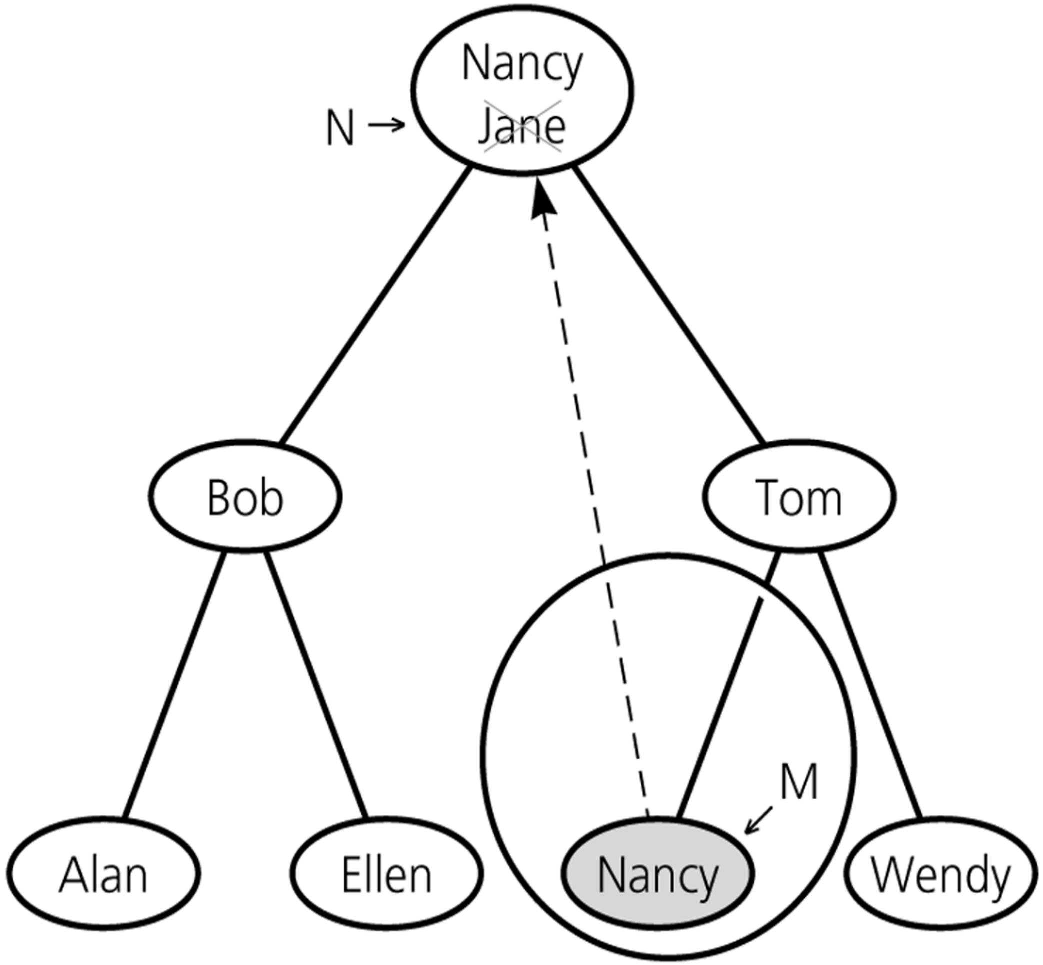
Deletion – Case3: A Node with two children

- Locate the inorder successor of the node.
- Copy the item in this node into the node which contains the item which will be deleted.
- Delete the node of the inorder successor.

Delete 40 (A node with two children)



Deletion – Case3: A Node with two children



Deletion routine for BST

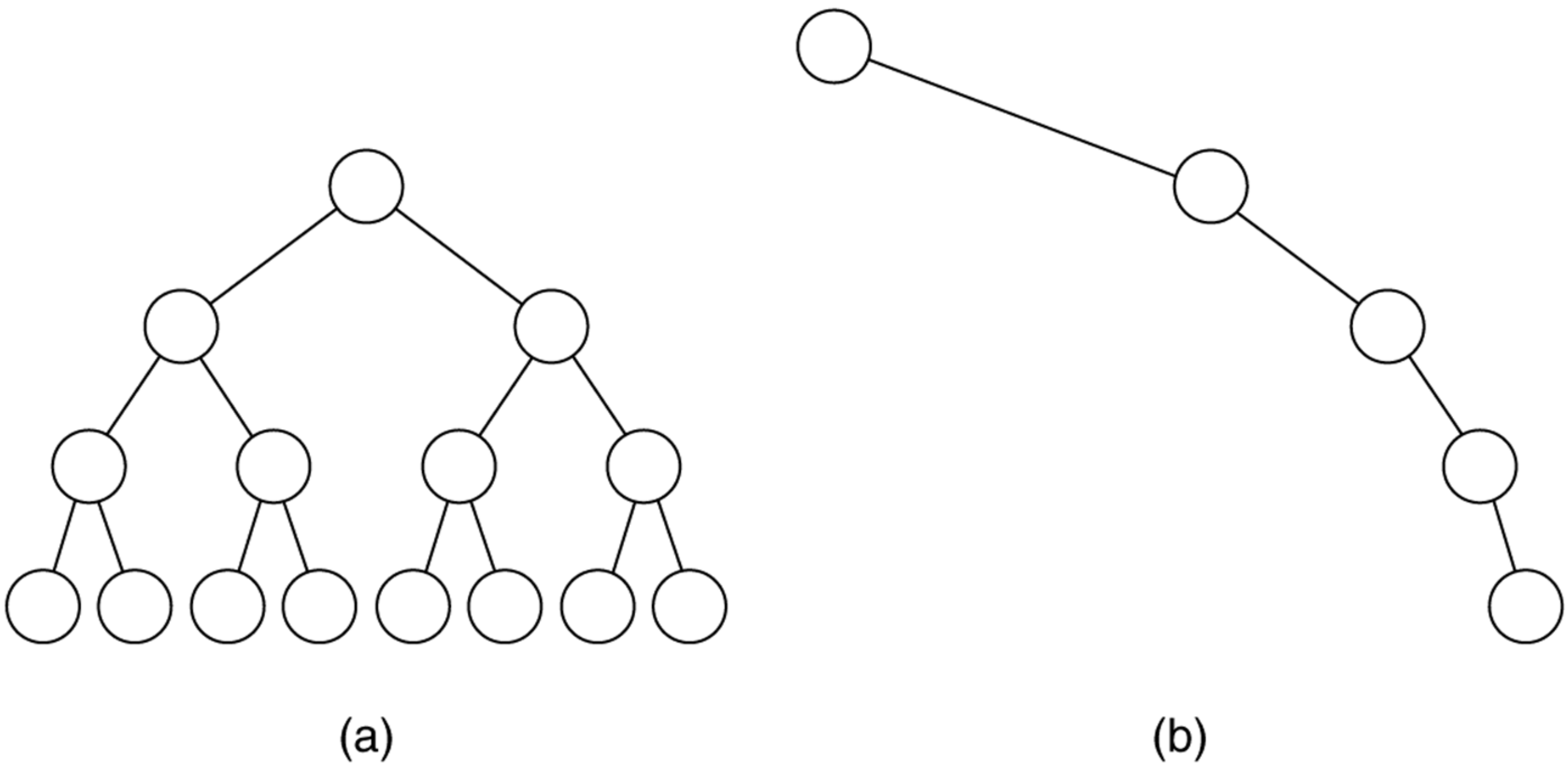
```
template <class Comparable>
void remove( const Comparable & x,
             BinaryNode<Comparable> * & t ) const
{
   if( t == NULL )
      return;   // Item not found; do nothing
   if( x < t->element )
      remove( x, t->left );
   else if( t->element < x )
      remove( x, t->right );
   else if( t->left != NULL && t->right != NULL {
       t->element = findMin( t->right )->element;
       remove( t->element, t->right );
   }
   else   {
      BinaryNode<Comparable> *oldNode = t;
      t = ( t->left != NULL ) ? t->left : t->right;
      delete oldNode;
    }
}
```

Analysis of BST Operations

- The cost of an operation is proportional to the depth of the last accessed node.
- The cost is logarithmic for a well-balanced tree, but it could be as bad as linear for a degenerate tree.
- In the best case we have logarithmic access cost, and in the worst case we have linear access cost.

Figure 19.19

(a) The balanced tree has a depth of log N; (b) the unbalanced tree has a depth of $N - 1$.

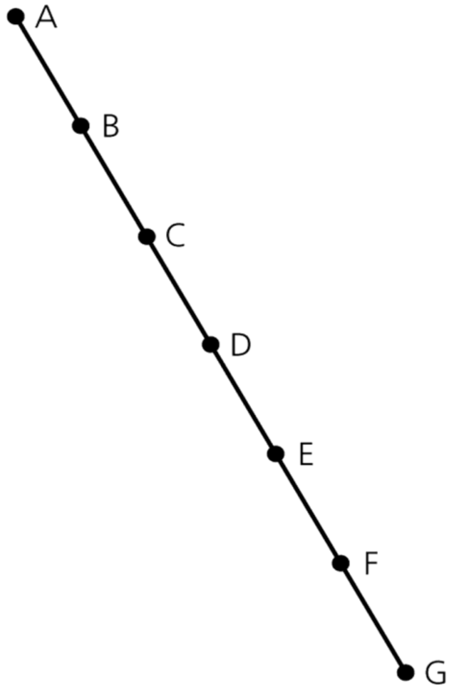


Maximum and Minimum Heights of a Binary Tree

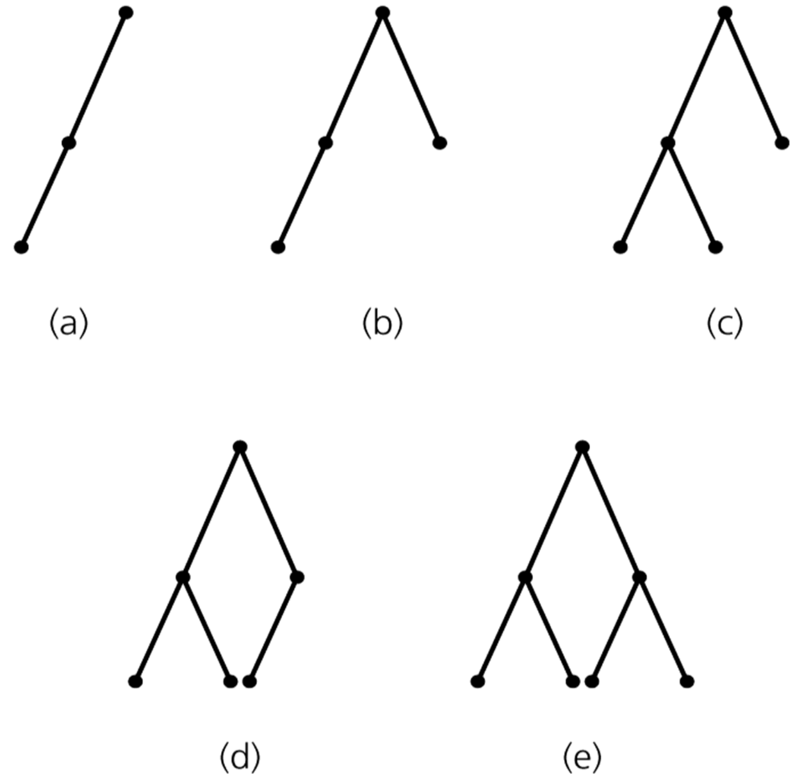
- The efficiency of most of the binary tree (and BST) operations depends on the height of the tree.
- The maximum number of key comparisons for retrieval, deletion, and insertion operations for BSTs is the height of the tree.

- The <u>maximum</u> of height of a binary tree with n nodes is n-1.
- Each level of a <u>minimum</u> height tree, except the last level, must contain as many nodes as possible.

Maximum and Minimum Heights of a Binary Tree

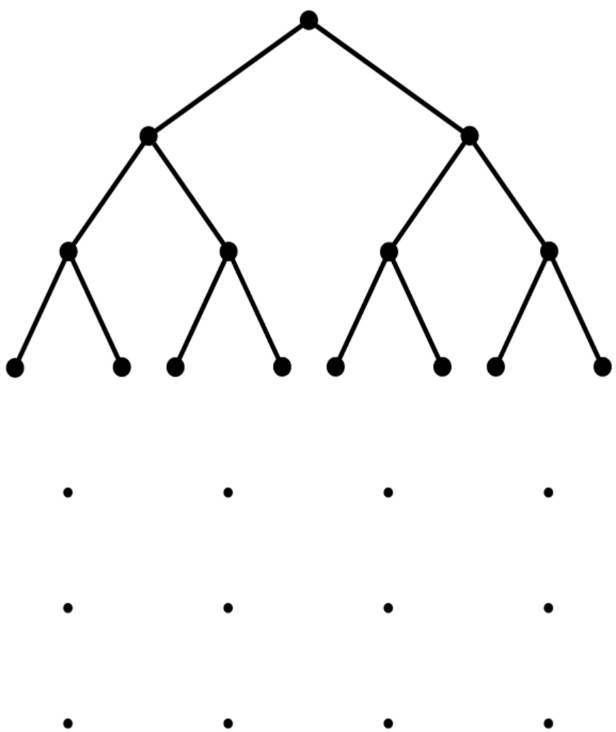


A maximum-height binary tree with seven nodes



Some binary trees of height 2

Counting the nodes in a full binary tree



Level	Number of nodes at this level	Number of nodes at this and previous levels
1	$1 = 2^0$	$1 = 2^1 - 1$
2	$2 = 2^1$	$3 = 2^2 - 1$
3	$4 = 2^2$	$7 = 2^3 - 1$
4	$8 = 2^3$	$15 = 2^4 - 1$
$\vdots$	$\vdots$	$\vdots$
h	2^{h-1}	$2^h - 1$

Some Height Theorems

Theorem 10-2: A full binary of height $h \geq 0$ has $2^{h+1}-1$ nodes.

Theorem 10-3: The maximum number of nodes that a binary tree of height h can have is $2^{h+1}-1$.

➔ We cannot insert a new node into a full binary tree without increasing its height.

Some Height Theorems

Theorem 10-4: The minimum height of a binary tree with n nodes is $\lfloor \log_2(n+1) \rfloor$.

Proof: Let h be the smallest integer such that $n \le 2^{h+1}-1$. We can establish following facts:

Fact 1 – A binary tree whose height is $\le$ h-1 has $\le$ n nodes.

- Otherwise h cannot be smallest integer in our assumption.

Fact 2 – There exists a complete binary tree of height h that has exactly n nodes.

- A full binary tree of height h-1 has 2^h-1 nodes.
- Since a binary tree of height h cannot have more than $2^{h+1}-1$ nodes.
- At level h, we will reach n nodes.

Fact 3 – The minimum height of a binary tree with n nodes is the smallest integer h such that $n \le 2^{h+1}-1$.

So, ➔ $2^h-1 < n \le 2^{h+1}-1$

➔ $2^h < n+1 \le 2^{h+1}$

➔ $h < \log_2(n+1) \le h+1$

Thus, ➔ $h = \lfloor \log_2(n+1) \rfloor$ is the minimum height of a binary tree with n nodes.

Minimum Height

- Complete trees and full trees have minimum height.
- The height of an n-node binary search tree ranges from $\lfloor \log_2(n+1) \rfloor$ to n-1.
- Insertion in search-key order produces a maximum-height binary search tree.
- Insertion in random order produces a near-minimum-height binary tree.
- That is, the height of an n-node binary search tree
  - *Best Case* – $\lfloor \log_2(n+1) \rfloor$ ➔ $O(\log_2 n)$
  - *Worst Case* – n-1 ➔ $O(n)$
  - *Average Case* – close to $\lfloor \log_2(n+1) \rfloor$ ➔ $O(\log_2 n)$
    - In fact, $1.39\log_2 n$

Average Height

Suppose we're inserting n items into an empty binary search tree to create a binary search tree with n nodes,

➔ How many different binary search trees with n nodes, and

➔ What are their probabilities,

There are n! different orderings of n keys.

But how many different binary search trees with n nodes?

n=0 ➔ 1 BST (empty tree)

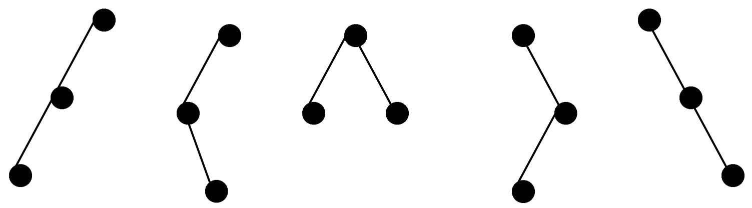
n=1 ➔ 1 BST (a binary tree with a single node)

n=2 ➔ 2 BSTs

n=3 ➔ 5 BSTs

Average Height (cont.)

n=3 ➔



Probabilities:	1/6	1/6	2/6	1/6	1/6
Insertion Order:	3,2,1	3,1,2	2,1,3 2,3,1	1,3,2	1,2,3

Order of Operations on BSTs

Operation	Average case	Worst case
Retrieval	O(log n)	O(n)
Insertion	O(log n)	O(n)
Deletion	O(log n)	O(n)
Traversal	O(n)	O(n)

Treesort

- We can use a binary search tree to sort an array.

```
treesort(inout anArray:ArrayType, in n:integer)
// Sorts n integers in an array anArray
// into ascending order
   Insert anArray's elements into a binary search
   tree bTree

   Traverse bTree in inorder. As you visit bTree's
  nodes,
   copy their data items into successive locations of
   anArray
```

Treesort Analysis

- Inserting an item into a binary search tree:
  - Worst Case: O(n)
  - Average Case: $O(\log_2 n)$
- Inserting n items into a binary search tree:
  - Worst Case: $O(n^2)$ ➔ $(1+2+...+n) = O(n^2)$
  - Average Case: $O(n*\log_2 n)$
- Inorder traversal and copy items back into array ➔ O(n)
- Thus, treesort is

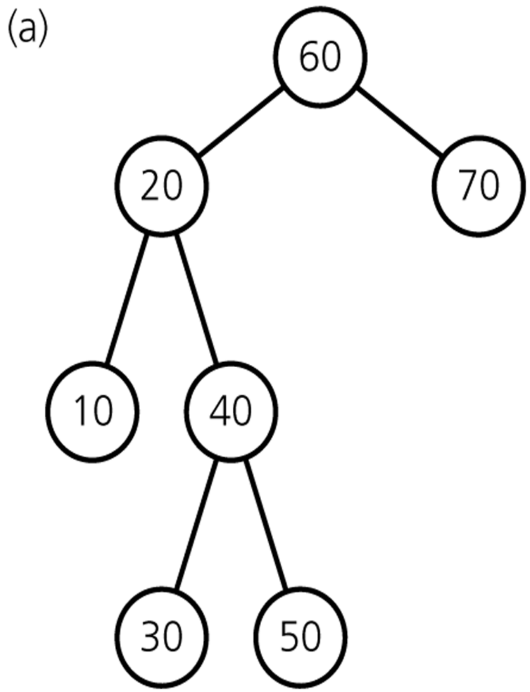
  ➔ **$O(n^2)$ in worst case**, and

  ➔ **$O(n*\log_2 n)$ in average case**.
- Treesort makes exactly the same comparisons of keys as quicksort when the pivot for each sublist is chosen to be the first key.

Saving a BST into a file, and restoring it to its original shape

- Save:
  - Use a preorder traversal to save the nodes of the BST into a file.
- Restore:
  - Start with an empty BST.
  - Read the nodes from the file one by one, and insert them into the BST.

Saving a BST into a file, and restoring it to its original shape



(b)

```
bst.searchTreeInsert(60);
bst.searchTreeInsert(20);
bst.searchTreeInsert(10);
bst.searchTreeInsert(40);
bst.searchTreeInsert(30);
bst.searchTreeInsert(50);
bst.searchTreeInsert(70);
```

Preorder: 60 20 10 40 30 50 70

Saving a BST into a file, and restoring it to a minimum-height BST

- Save:
  - Use an inorder traversal to save the nodes of the BST into a file. The saved nodes will be in ascending order.
  - Save the number of nodes (n) in somewhere.

- Restore:
  - Read the number of nodes (n).
  - Start with an empty BST.
  - Read the nodes from the file one by one to create a minimum-height binary search tree.

Building a minimum-height BST

```
readTree(out treePtr:TreeNodePtr, in n:integer)
// Builds a minimum-height binary search tree fro n sorted
// values in a file. treePtr will point to the tree’s root.

if (n>0) {
    // construct the left subtree
    treePtr = pointer to new node with NULL child pointers
    readTree(treePtr->leftChildPtr, n/2)

    // get the root
    Read item from file into treePtr->item

    // construct the right subtree
    readTree(treePtr->rightChildPtr, (n-1)/2)
  }
```

A full tree saved in a file by using inorder traversal