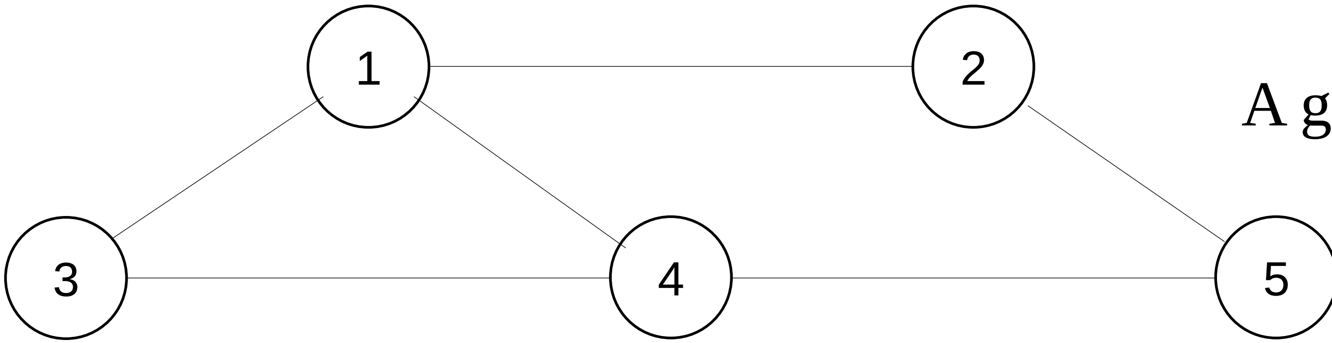
GRAPHS – Definitions

- A **graph** G = (V, E) consists of
  - a set of ***vertices***, V, and
  - a set of ***edges***, E, where each edge is a pair (v,w) s.t. $v,w \in V$
- Vertices are sometimes called ***nodes***, edges are sometimes called ***arcs***.
- If the edge pair is ordered then the graph is called a **directed graph (**also called *digraphs)* .
- We also call a normal graph (which is not a directed graph) an ***undirected graph***.
  - When we say graph we mean that it is an undirected graph.

Graph – Definitions

- Two vertices of a graph are ***adjacent*** if they are joined by an edge.
- Vertex w is ***adjacent to*** v iff $(v,w) \in E$.
  - In an undirected graph with edge (v, w) and hence (w,v) w is adjacent to v and v is adjacent to w.
- A ***path*** between two vertices is a sequence of edges that begins at one vertex and ends at another vertex.
  - i.e. $w_1, w_2, \ldots, w_N$ is a path if $(w_i, w_{i+1}) \in E$ for $1 \leq i \leq. N-1$
- A ***simple path*** passes through a vertex only once.
- A ***cycle*** is a path that begins and ends at the same vertex.
- A ***simple cycle*** is a cycle that does not pass through other vertices more than once.

Graph – An Example



A graph G (undirected)

The graph G= (V,E) has 5 vertices and 6 edges:

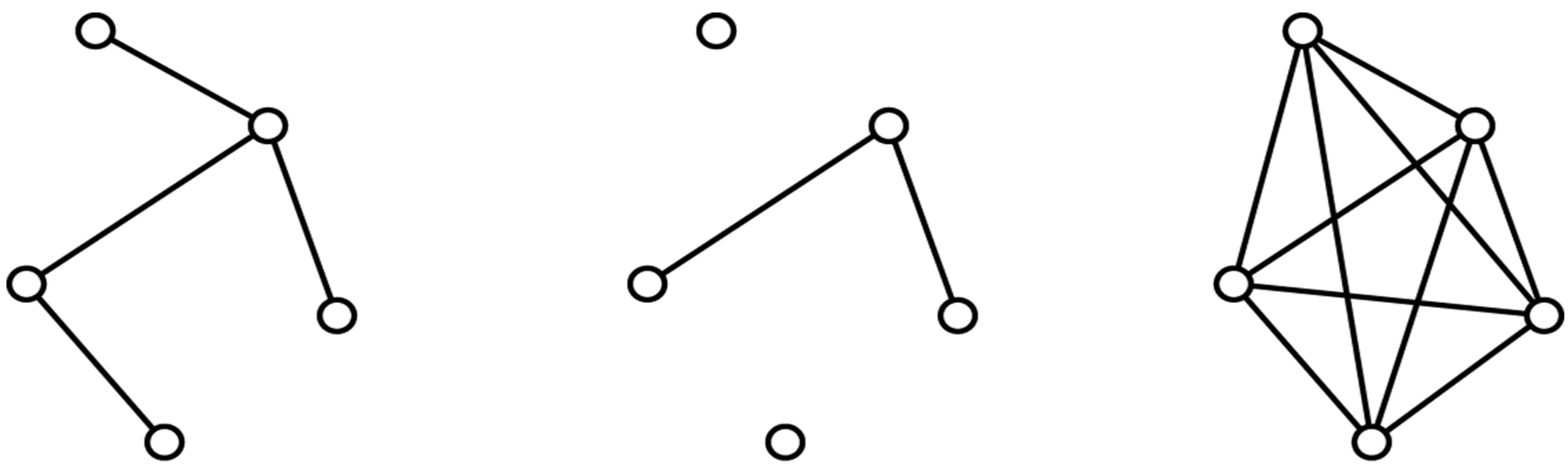
V = {1,2,3,4,5}

E = { (1,2),(1,3),(1,4),(2,5),(3,4),(4,5), (2,1),(3,1),(4,1),(5,2),(4,3),(5,4) }

- *Adjacent:*
  1 and 2 are adjacent -- 1 is adjacent to 2 and 2 is adjacent to 1
- *Path:*
  1,2,5 (a simple path), 1,3,4,1,2,5 (a path but not a simple path)
- *Cycle:*
  1,3,4,1 (a simple cycle), 1,3,4,1,4,1 (cycle, but not simple cycle)

Graph -- Definitions

- A ***connected graph*** has a path between each pair of distinct vertices.
- A ***complete graph*** has an edge between each pair of distinct vertices.
  - A complete graph is also a connected graph. But a connected graph may not be a complete graph.



(a) **connected** (b) **disconnected** (c) **complete**

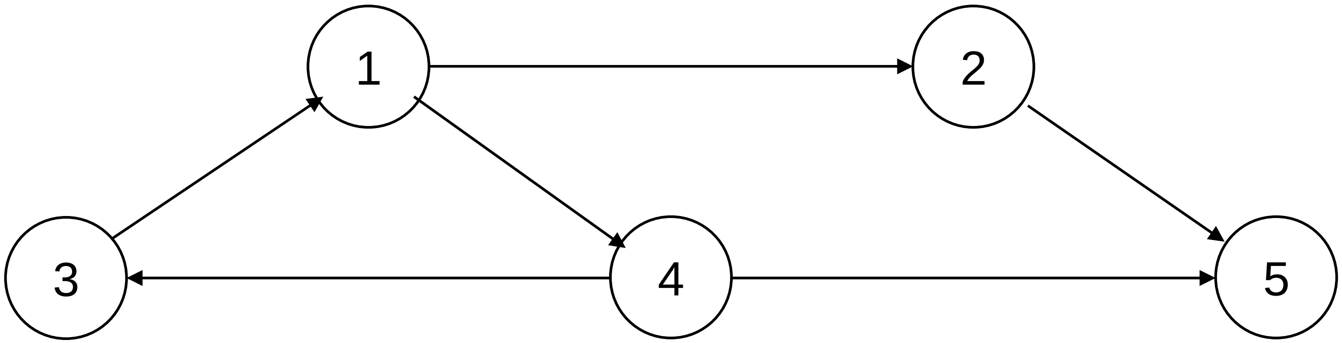
Directed Graphs

- If the edge pair is ordered then the graph is called a **directed graph (**also called *digraphs)* .
- Each edge in a directed graph has a direction, and each edge is called a ***directed edge***.
- Definitions given for undirected graphs apply also to directed graphs, with changes that account for direction.
- Vertex w is ***adjacent to*** v iff $(v,w) \in E$.
  - i.e. There is a direct edge from v to w
  - w is ***successor*** of v
  - v is ***predecessor*** of w
- A ***directed path*** between two vertices is a sequence of directed edges that begins at one vertex and ends at another vertex.
  - i.e. $w_1, w_2, \ldots, w_N$ is a path if $(w_i, w_{i+1}) \in E$ for $1 \leq i \leq$. N-1

Directed Graphs

- A **cycle** in a directed graph is a path of length at least 1 such that $w_1 = w_N$.
  - This cycle is simple if the path is simple.
  - For undirected graphs, the edges must be distinct
- A **directed acyclic graph** (*DAG*) is a type of directed graph having no cycles.
- An undirected graph is **connected** if there is a path from every vertex to every other vertex.
- A directed graph with this property is called **strongly connected**.
  - If a directed graph is not strongly connected, but the underlying graph (without direction to arcs) is connected then the graph is **weakly connected**.

Directed Graph – An Example



The graph G= (V,E) has 5 vertices and 6 edges:

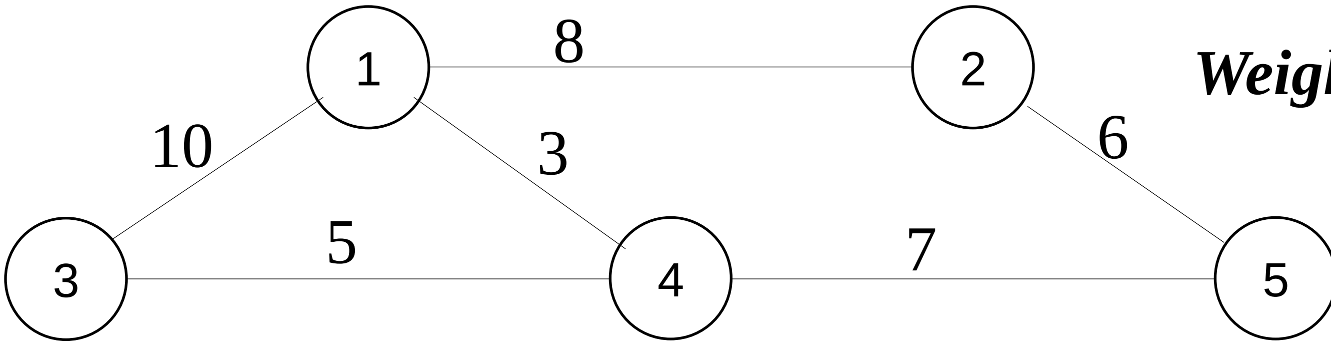
V = {1,2,3,4,5}

E = { (1,2),(1,4),(2,5),(4,5),(3,1),(4,3) }

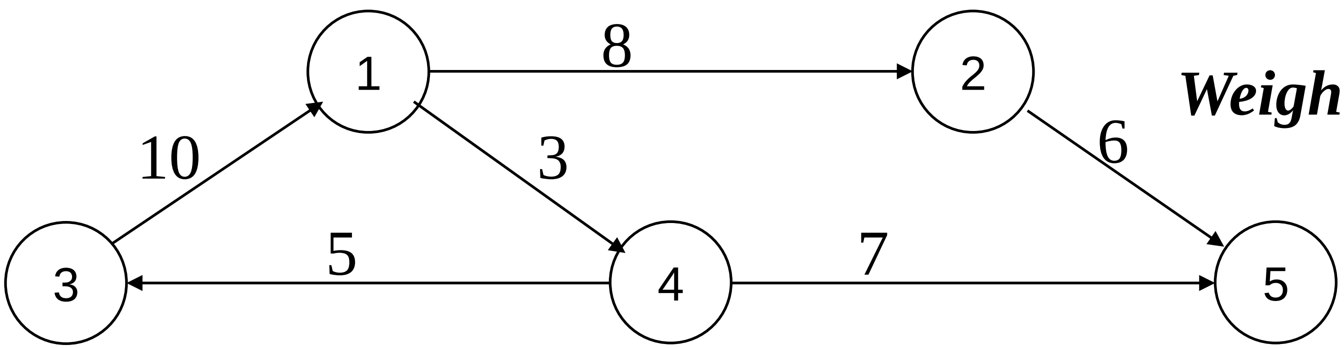
- *Adjacent:*
  2 is adjacent to 1, but 1 is NOT adjacent to 2
- *Path:*
  1,2,5 (a directed path),
- *Cycle:*
  1,4,3,1 (a directed cycle),

Weighted Graph

- We can label the edges of a graph with numeric values, the graph is called a ***weighted graph***.



Weighted (Undirected) Graph



Weighted Directed Graph

Graph Implementations

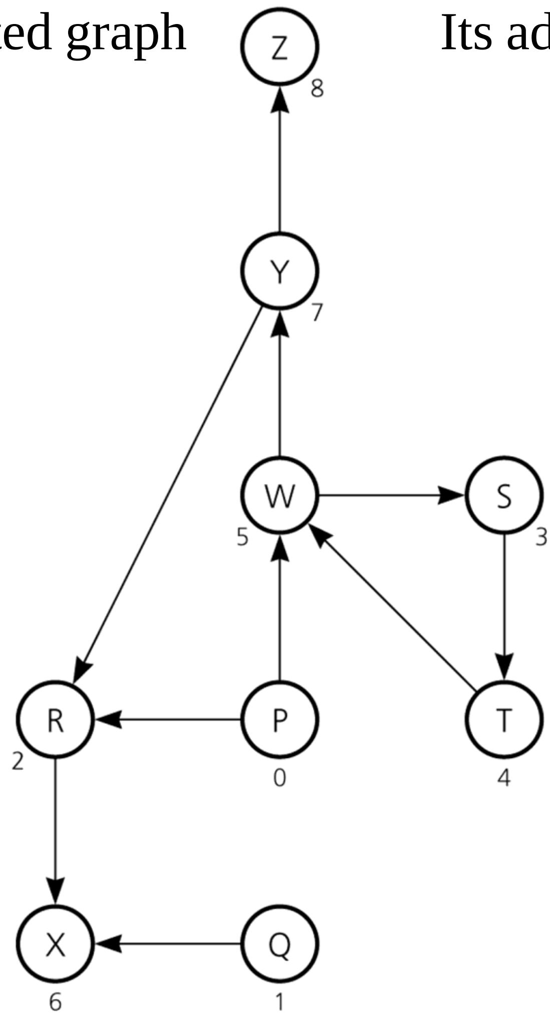
- The two most common implementations of a graph are:
  - ***Adjacency Matrix***
    - A two dimensional array
  - ***Adjacency List***
    - For each vertex we keep a list of adjacent vertices

Adjacency Matrix

- An ***adjacency matrix*** for a graph with *n* vertices numbered 0,1,...,n-1 is an *n* by *n* array *matrix* such that *matrix[i][j]* is 1 (true) if there is an edge from vertex *i* to vertex *j*, and 0 (false) otherwise.
- When the graph is *weighted*, we can let *matrix[i][j]* be the weight that labels the edge from vertex *i* to vertex *j*, instead of simply 1, and let *matrix[i][j]* equal to ∞ instead of 0 when there is no edge from vertex *i* to vertex *j*.
- Adjacency matrix for an undirected graph is symmetrical.
  - i.e. *matrix[i][j]* is equal to *matrix[j][i]*
- Space requirement $O(|V|^2)$
- Acceptable if the graph is dense.

Adjacency Matrix – Example1

A directed graph

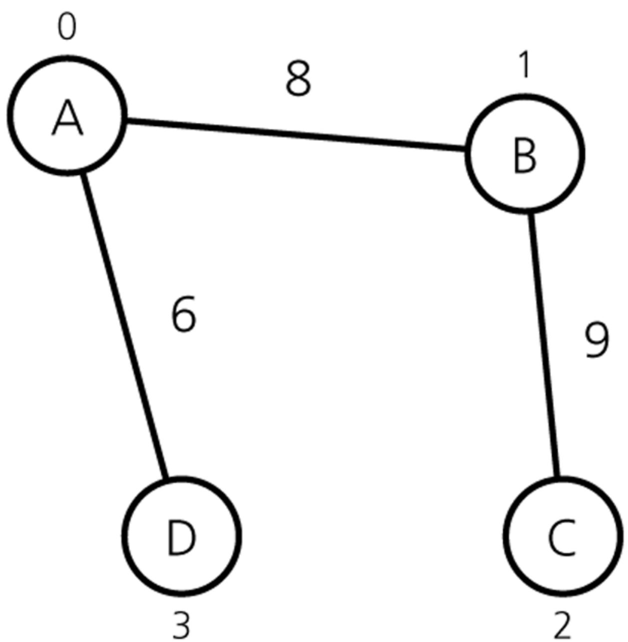


Its adjacency matrix

		0	1	2	3	4	5	6	7	8
		P	Q	R	S	T	W	X	Y	Z
0	P	0	0	1	0	0	1	0	0	0
1	Q	0	0	0	0	0	0	1	0	0
2	R	0	0	0	0	0	0	1	0	0
3	S	0	0	0	0	1	0	0	0	0
4	T	0	0	0	0	0	1	0	0	0
5	W	0	0	0	1	0	0	0	1	0
6	X	0	0	0	0	0	0	0	0	0
7	Y	0	0	1	0	0	0	0	0	1
8	Z	0	0	0	0	0	0	0	0	0

Adjacency Matrix – Example2

An Undirected Weighted Graph



Its Adjacency Matrix

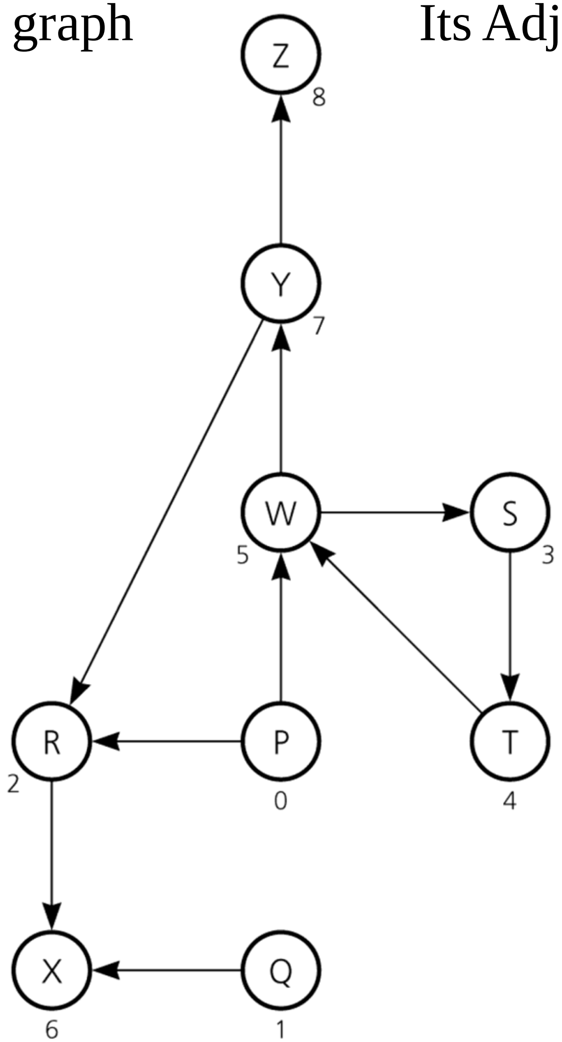
		0 A	1 B	2 C	3 D
0	A	∞	8	∞	6
1	B	8	∞	9	∞
2	C	∞	9	∞	∞
3	D	6	∞	∞	∞

Adjacency List

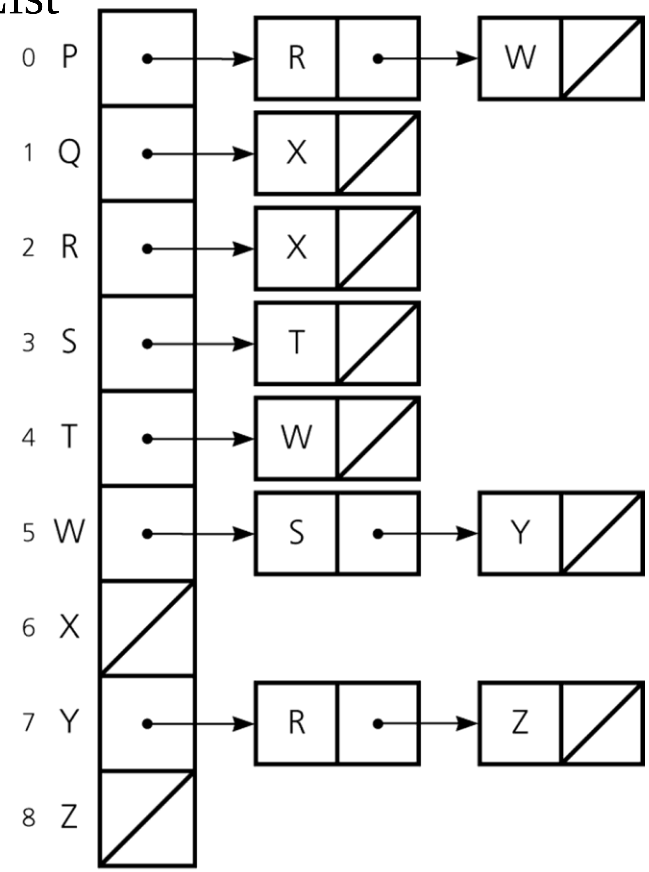
- An ***adjacency list*** for a graph with *n* vertices numbered 0,1,...,n-1 consists of *n* linked lists. The i^{th} linked list has a node for vertex *j* if and only if the graph contains an edge from vertex *i* to vertex *j*.
- Adjacency list is a better solution if the graph is sparse.
- Space requirement is O(|E| + |V|), which is linear in the size of the graph.
- In an undirected graph each edge (v,w) appears in two lists.
  - Space requirement is doubled.

Adjacency List – Example1

A directed graph

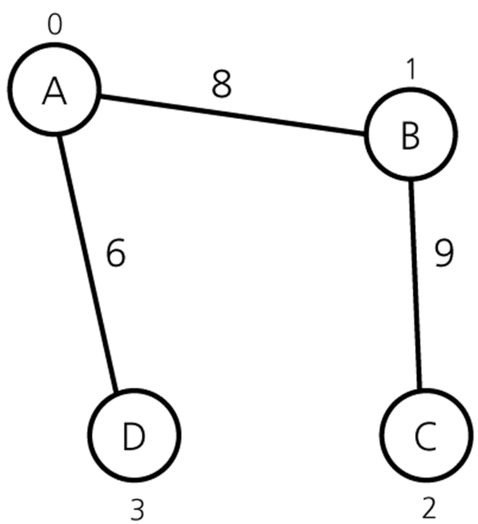


Its Adjacency List

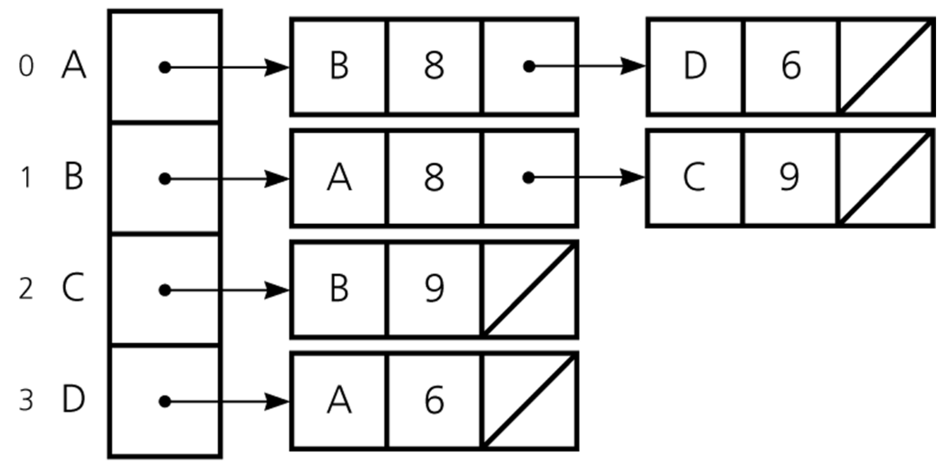


Adjacency List – Example2

An Undirected Weighted Graph



Its Adjacency List



Adjacency Matrix vs Adjacency List

- Two common graph operations:
  1. Determine whether there is an edge from vertex i to vertex j.
  2. Find all vertices adjacent to a given vertex i.

- An adjacency matrix supports operation 1 more efficiently.
- An adjacency list supports operation 2 more efficiently.

- An adjacency list often requires less space than an adjacency matrix.
  - Adjacency Matrix: Space requirement is $O(|V|^2)$
  - Adjacency List : Space requirement is $O(|E| + |V|)$, which is linear in the size of the graph.
  - Adjacency matrix is better if the graph is dense (too many edges)
  - Adjacency list is better if the graph is sparse (few edges)

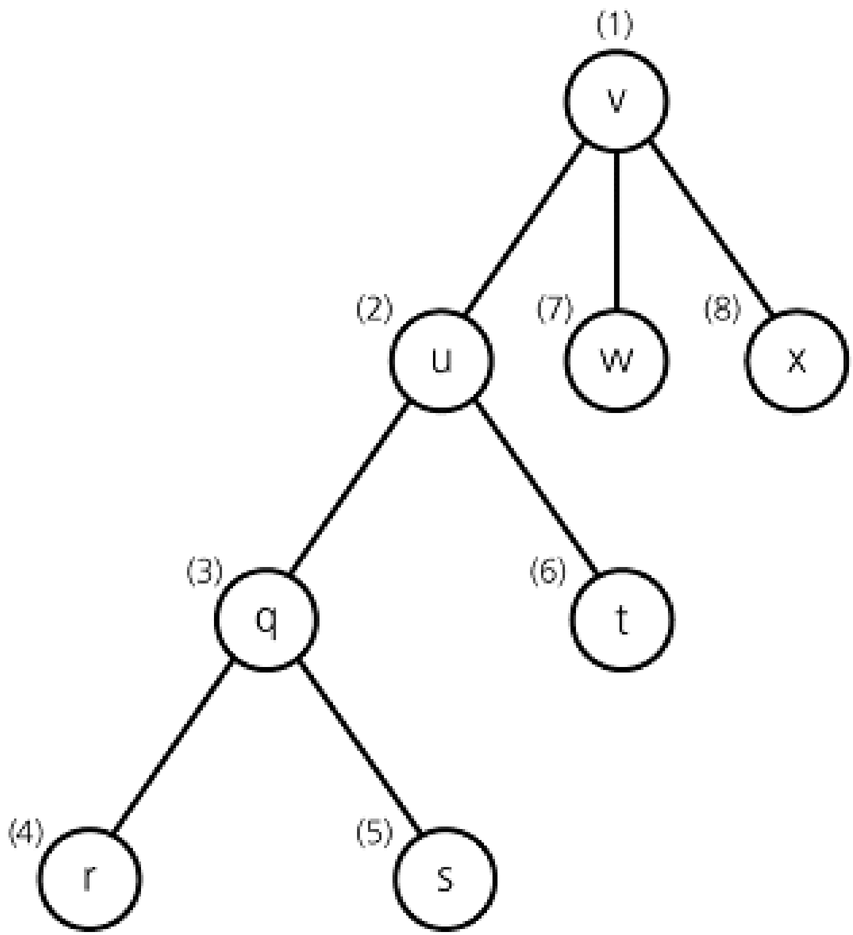
Graph Traversals

- A *graph-traversal* algorithm starts from a vertex v, visits all of the vertices that can be reachable from the vertex v.
- A graph-traversal algorithm visits all vertices if and only if the graph is connected.
- A connected component is the subset of vertices visited during a traversal algorithm that begins at a given vertex.
- A graph-traversal algorithm must mark each vertex during a visit and must never visit a vertex more than once.
  – Thus, if a graph contains a cycle, the graph-traversal algorithm can avoid infinite loop.
- We look at two graph-traversal algorithms:
  – ***Depth-First Traversal***
  – ***Breadth-First Traversal***

Depth-First Traversal

- For a given vertex v, the ***depth-first traversal*** algorithm proceeds along a path from v as deeply into the graph as possible before backing up.
- That is, after visiting a vertex v, the ***depth-first traversal*** algorithm visits (if possible) an unvisited adjacent vertex to vertex v.
- The depth-first traversal algorithm does not completely specify the order in which it should visit the vertices adjacent to v.
  - We may visit the vertices adjacent to v in sorted order.

Depth-First Traversal – Example



- A depth-first traversal of the graph starting from vertex v.

- Visit a vertex, then visit a vertex adjacent to that vertex.

- If there is no unvisited vertex adjacent to visited vertex, back up to the previous step.

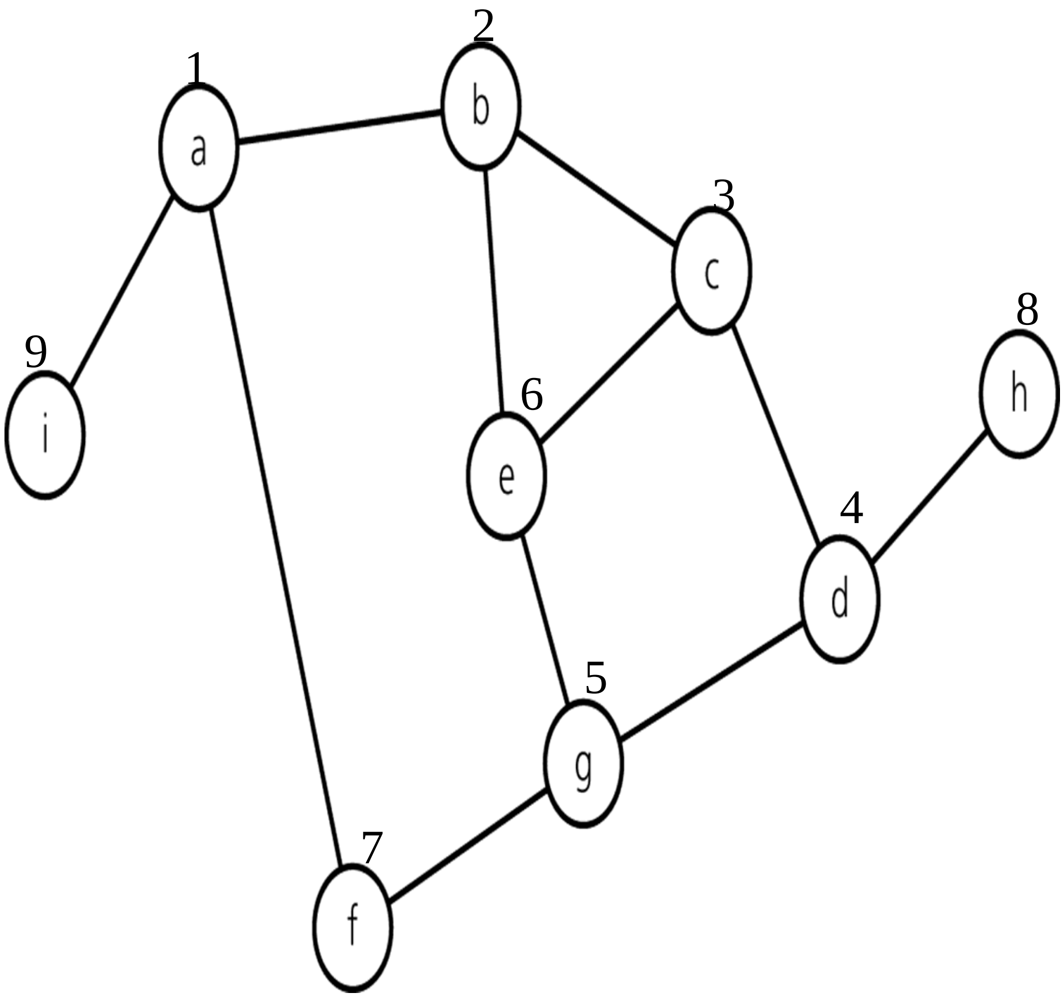
Recursive Depth-First Traversal Algorithm

```
dft(in v:Vertex) {
// Traverses a graph beginning at vertex v
// by using depth-first strategy
// Recursive Version
  Mark v as visited;
  for (each unvisited vertex u adjacent to v)
      dft(u)
}
```

Iterative Depth-First Traversal Algorithm

```
dft(in v:Vertex) {
// Traverses a graph beginning at vertex v
// by using depth-first strategy: Iterative Version
   s.createStack();
   // push v into the stack and mark it
   s.push(v);
   Mark v as visited;
   while (!s.isEmpty()) {
      if (no unvisited vertices are adjacent to the vertex on
          the top of stack)
         s.pop();  // backtrack
      else {
         Select an unvisited vertex u adjacent to the vertex
            on the top of the stack;
         s.push(u);
         Mark u as visited;
      }
   }
}
```

Trace of Iterative DFT – starting from vertex a

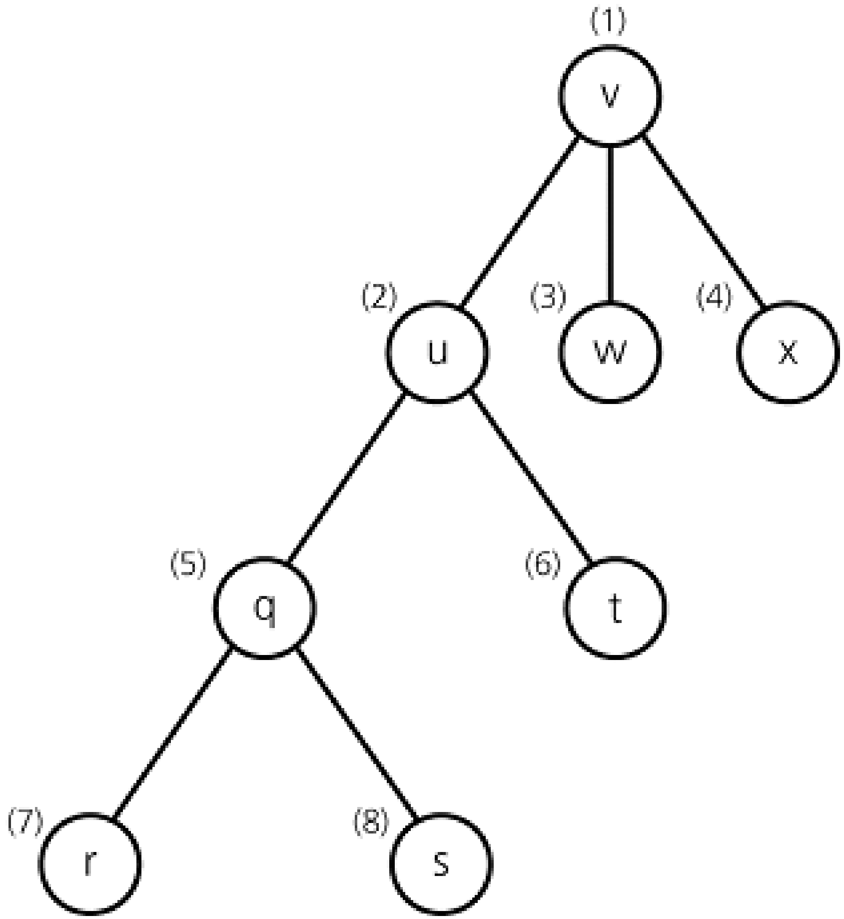


Node visited	Stack (bottom to top)
a	a
b	a b
c	a b c
d	a b c d
g	a b c d g
e	a b c d g e
(backtrack)	a b c d g
f	a b c d g f
(backtrack)	a b c d g
(backtrack)	a b c d
h	a b c d h
(backtrack)	a b c d
(backtrack)	a b c
(backtrack)	a b
(backtrack)	a
i	a i
(backtrack)	a
(backtrack)	*(empty)*

Breath-First Traversal

- After visiting a given vertex v, the breadth-first traversal algorithm visits every vertex adjacent to v that it can before visiting any other vertex.
- The breath-first traversal algorithm does not completely specify the order in which it should visit the vertices adjacent to v.
  - We may visit the vertices adjacent to v in sorted order.

Breath-First Traversal – Example



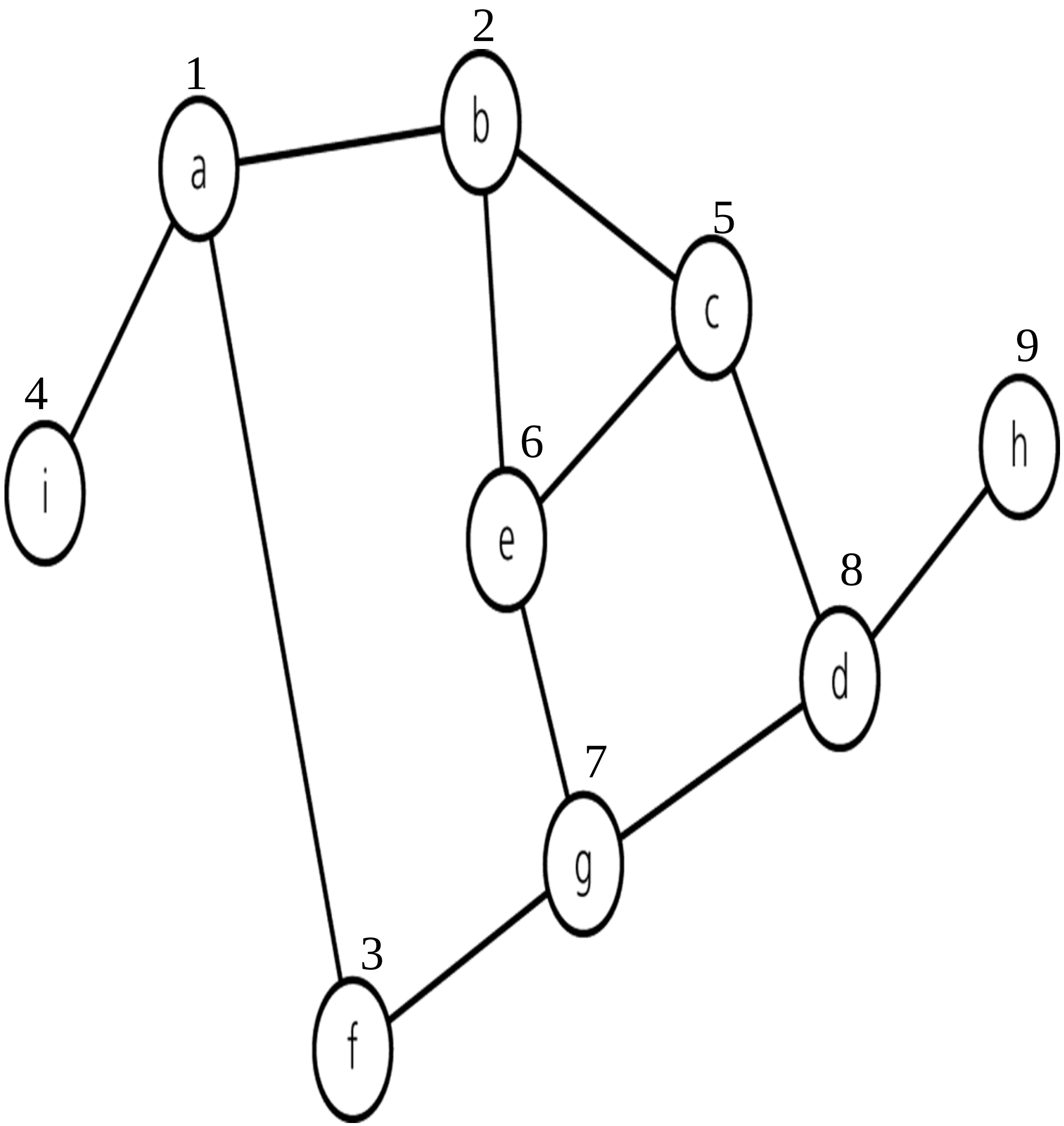
- A breath-first traversal of the graph starting from vertex v.

- Visit a vertex, then visit all vertices adjacent to that vertex.

Iterative Breath-First Traversal Algorithm

```
bft(in v:Vertex) {
// Traverses a graph beginning at vertex v
// by using breath-first strategy: Iterative Version
   q.createQueue();
   // add v to the queue and mark it
   q.enqueue(v);
   Mark v as visited;
   while (!q.isEmpty()) {
      q.dequeue(w);
      for (each unvisited vertex u adjacent to w) {
         Mark u as visited;
         q.enqueue(u);
      }
   }
}
```

Trace of Iterative BFT – starting from vertex a



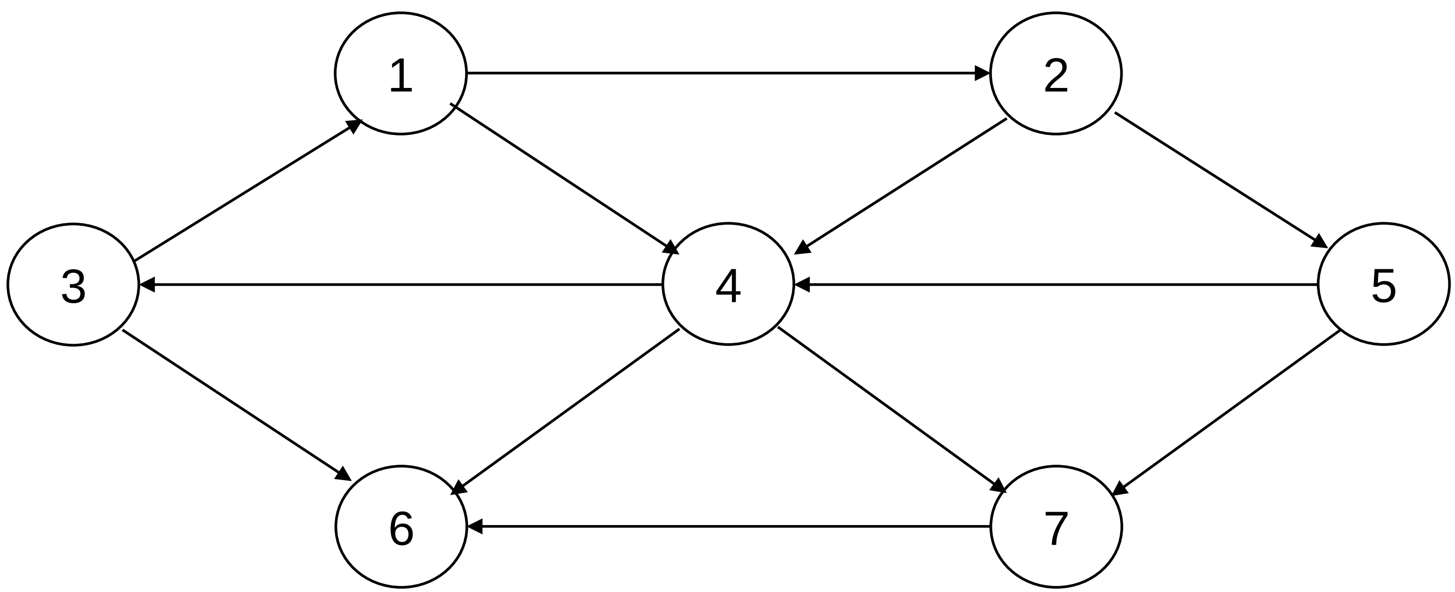
Node visited	Queue (front to back)
a	a
	(empty)
b	b
f	b f
i	b f i
	f i
c	f i c
e	f i c e
	i c e
g	i c e g
	c e g
	e g
d	e g d
	g d
	d
	(empty)
h	h
	(empty)

Some Graph Algorithms

- Shortest Path Algorithms
  - Unweighted shortest paths
  - Weighted shortest paths (Dijkstra’s Algorithm)
- Topological sorting
- Network Flow Problems
- Minimum Spanning Tree
- Depth-first search Applications

Unweighted Shortest-Path problem

- *Find the shortest path (measured by number of edges) from a designated vertex S to every vertex.*



Algorithm

1. Start with an initial node s.
   - Mark the distance of s to s, D_s as 0.
   - Initially $D_i = \infty$ for all $i \neq s$.
2. Traverse all nodes starting from s as follows:
   1. If the node we are currently visiting is v, for all *w* that are adjacent to v:
      - Set $D_w = D_v + 1$ if $D_w = \infty$.
   2. Repeat step 2.1 with another vertex u that has not been visited yet, such that $D_u = D_v$ (if any).
   3. Repeat step 2.1 with another unvisited vertex u that satisfies $D_u = D_v + 1$.(if any)

Figure 14.21A

Searching the graph in the unweighted shortest-path computation. The darkest-shaded vertices have already been completely processed, the lightest-shaded vertices have not yet been used as *v*, and the medium-shaded vertex is the current vertex, *v*. The stages proceed left to right, top to bottom, as numbered *(continued).*

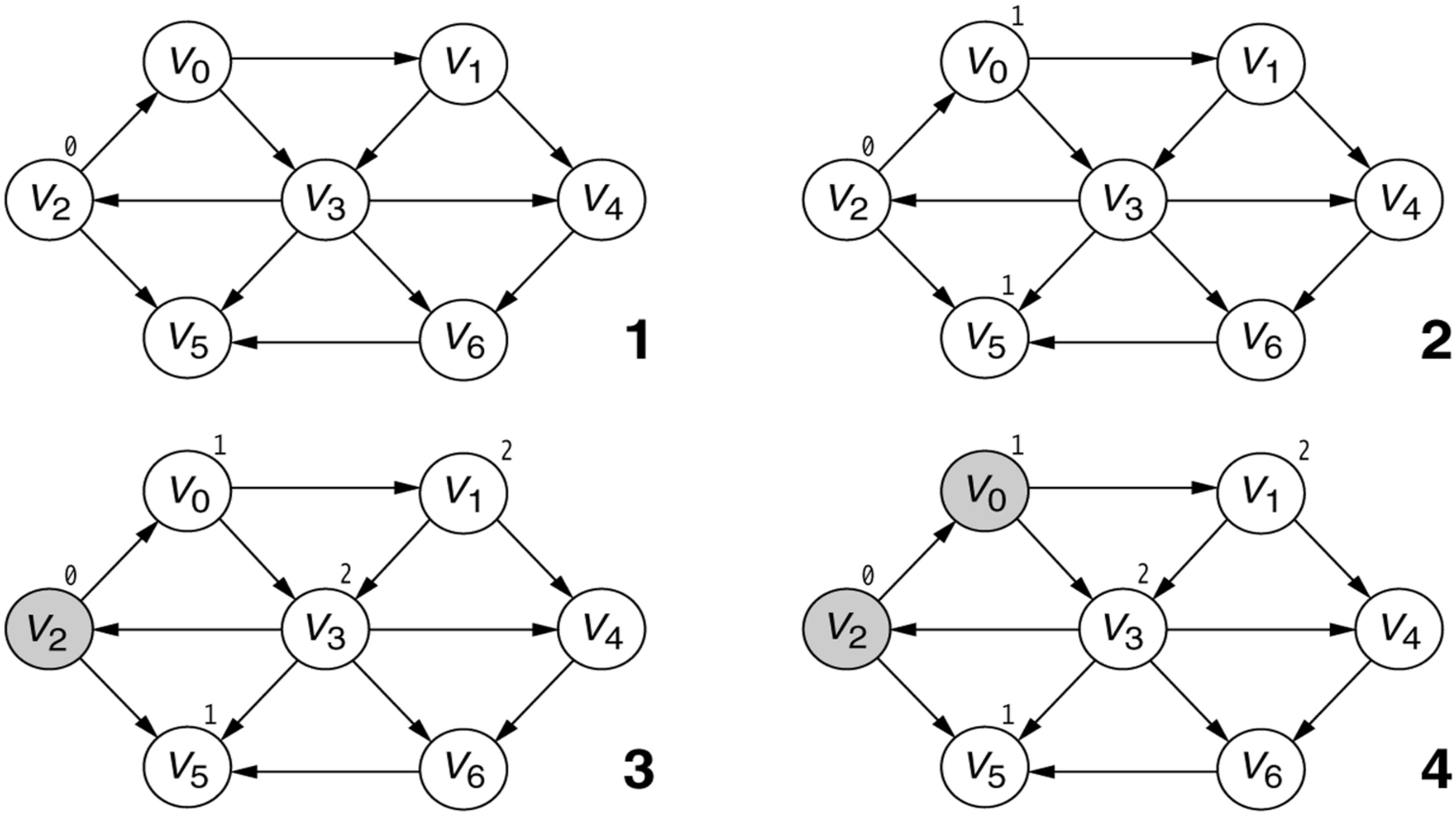
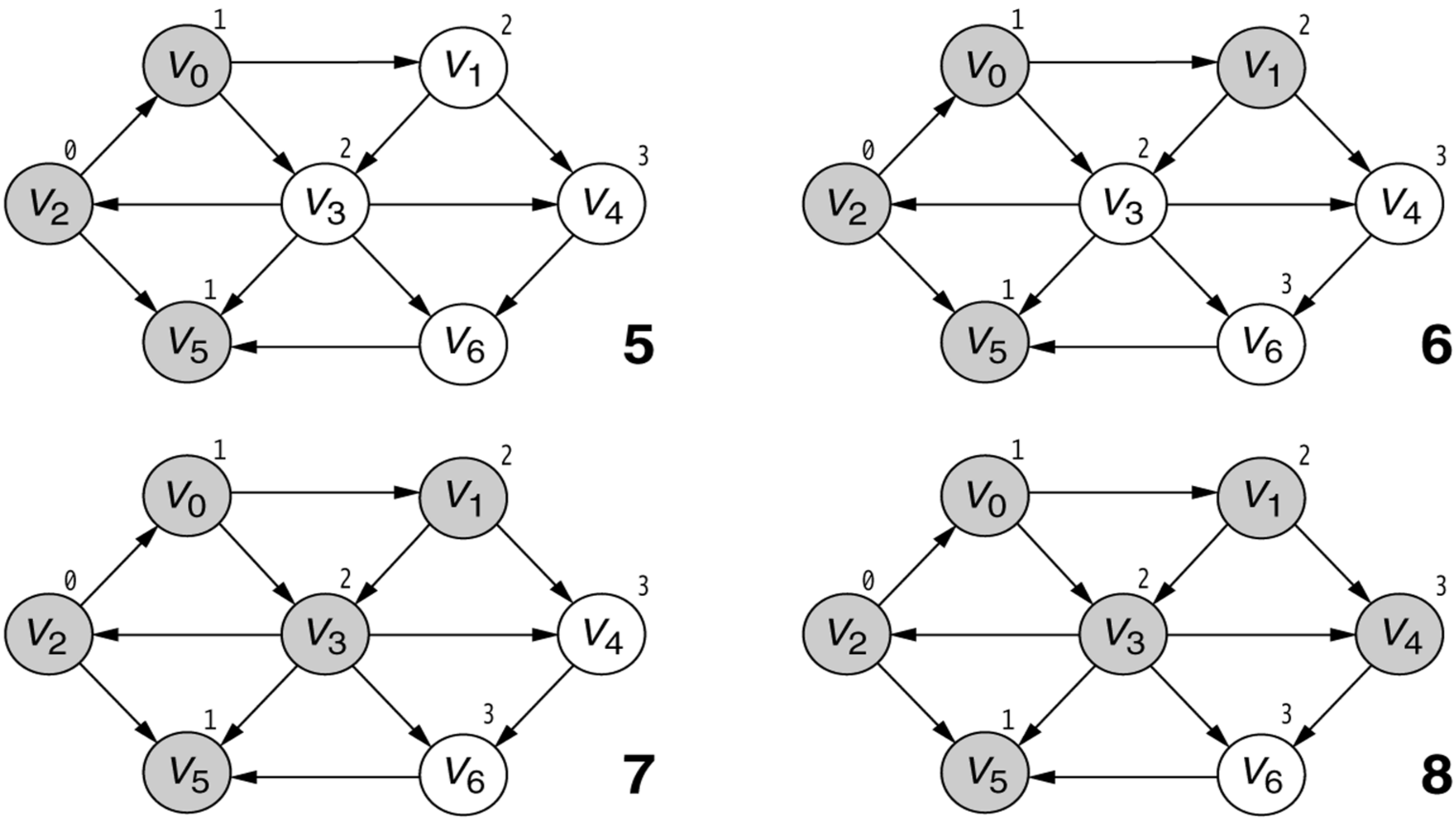


Figure 14.21B

Searching the graph in the unweighted shortest-path computation. The darkest-shaded vertices have already been completely processed, the lightest-shaded vertices have not yet been used as *v*, and the medium-shaded vertex is the current vertex, *v*. The stages proceed left to right, top to bottom, as numbered.



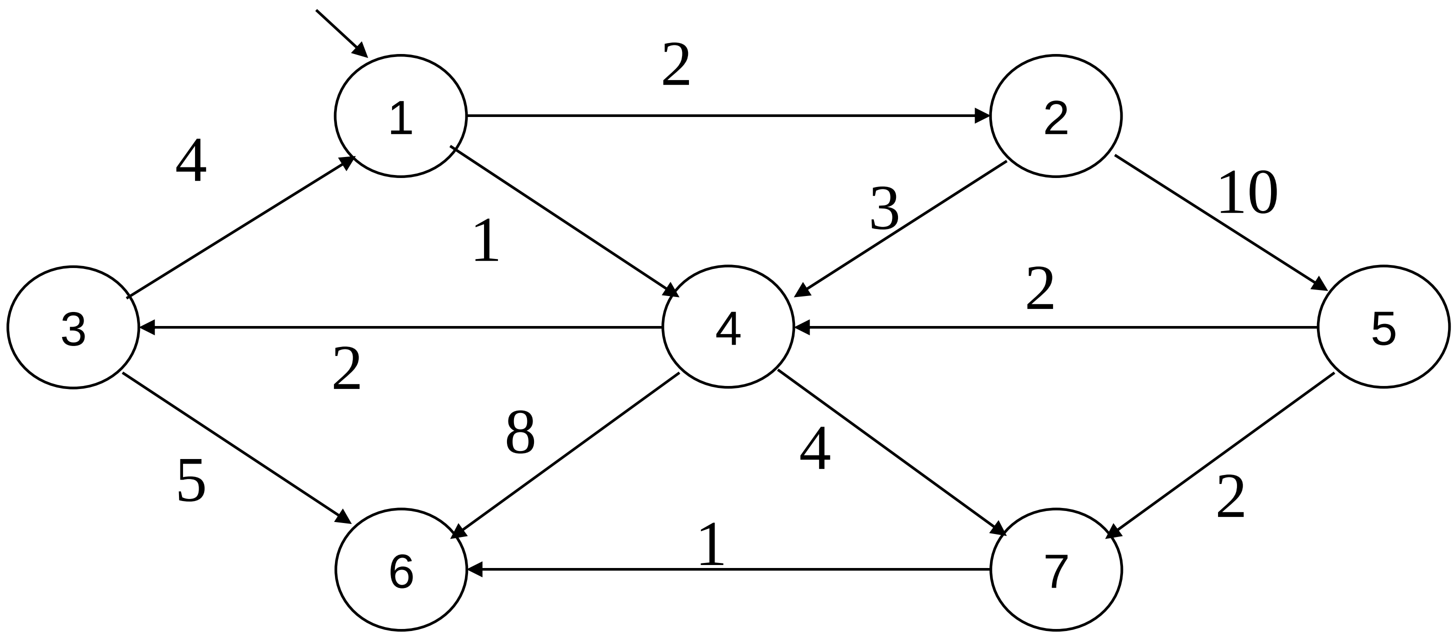
Unweighted shortest path algorithm

```
void Graph::unweighted_shortest_paths(vertex s)
{
   Queue q(NUM_VERTICES);
   Vertex v,w;

   q.enqueue(s);
   s.dist = 0;
   while (!q.isEmpty())
   {
        v= q.dequeue();
        v.known = true; // not needed anymore
        for each w adjacent to v
               if (w.dist == INFINITY)
               {
                      w.dist = v.dist + 1;
                      w.path = v;
                      q.enqueue(w);
               }
   }
}
```

Weighted Shortest-path Problem

- *Find the shortest path (measured by total cost) from a designated vertex S to every vertex. All edge costs are nonnegative.*

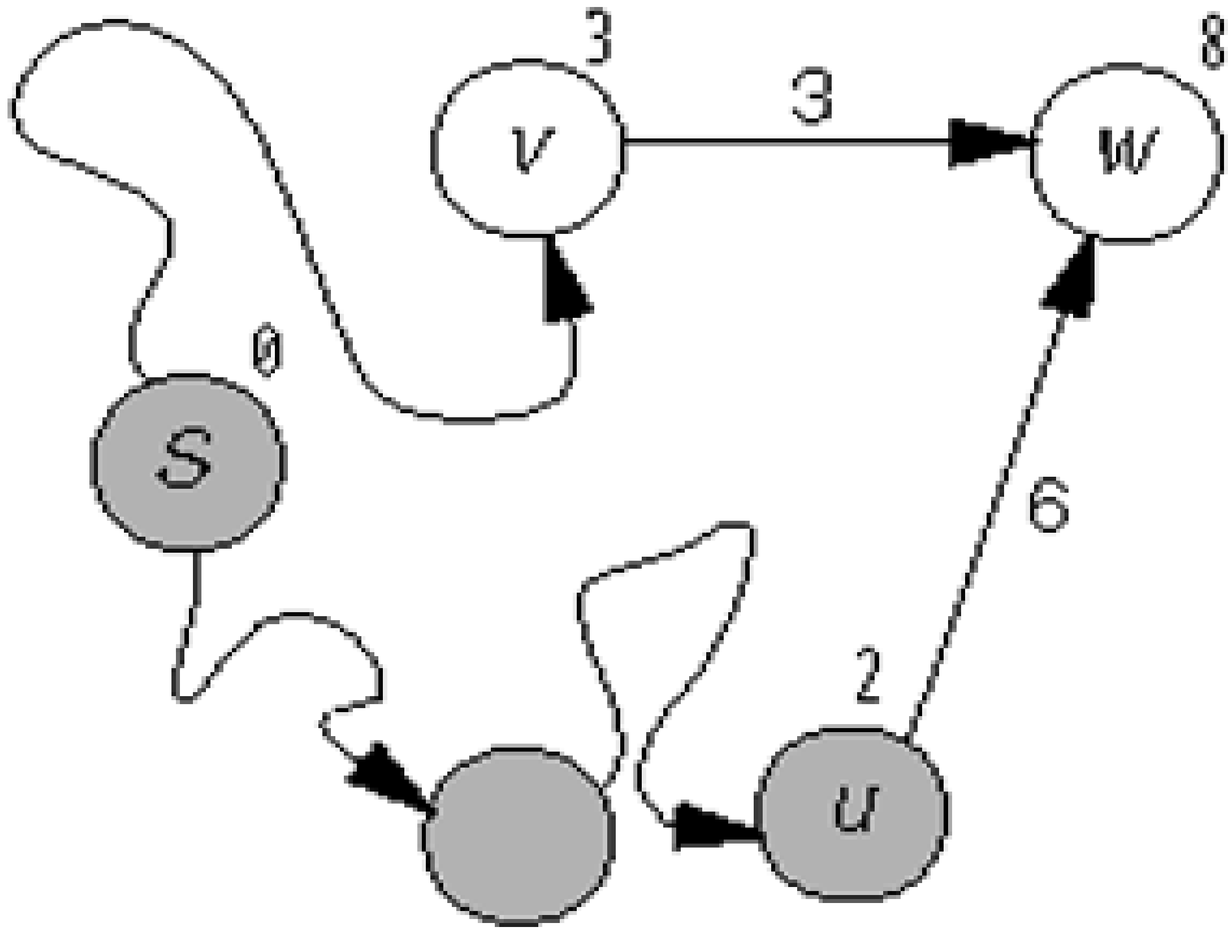


Weighted Shortest-path Problem

- The method used to solve this problem is known as Dijkstra’s algorithm.
  - An example of a greedy algorithm
  - Use the local optimum at each step
- Solution is similar to the solution of unweighted shortest path problem.
- The following issues must be examined:
  - How do we adjust D_w?
  - How do we find the vertex v to visit next?

Figure 14.23

The eyeball is at v and w is adjacent, so D_w should be lowered to 6.



Dijkstra's algorithm

- The algorithm proceeds in stages.
- At each stage, the algorithm
  - selects a vertex v, which has the smallest distance D_v among all the *unknown* vertices, and
  - declares that the shortest path from s to v is *known*.
  - then for the adjacent nodes of v (which are denoted as w) D_w is updated with new distance information
- How do we change D_w?
  - If its current value is larger than $D_v + c_{v,w}$ we change it.

Figure 14.25A

Stages of Dijkstra’s algorithm. The conventions are the same as those in Figure 14.21 (*continued*).

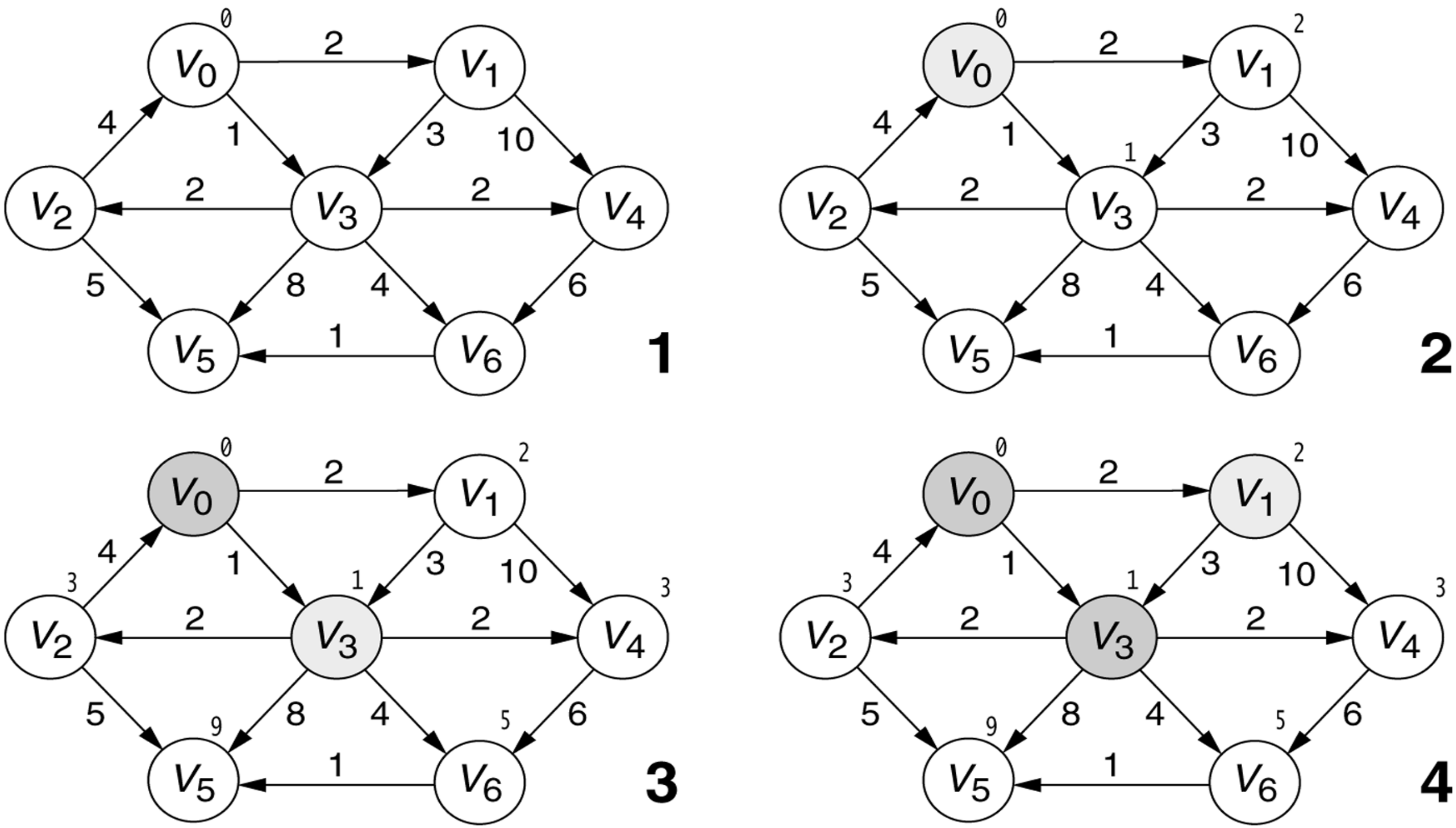
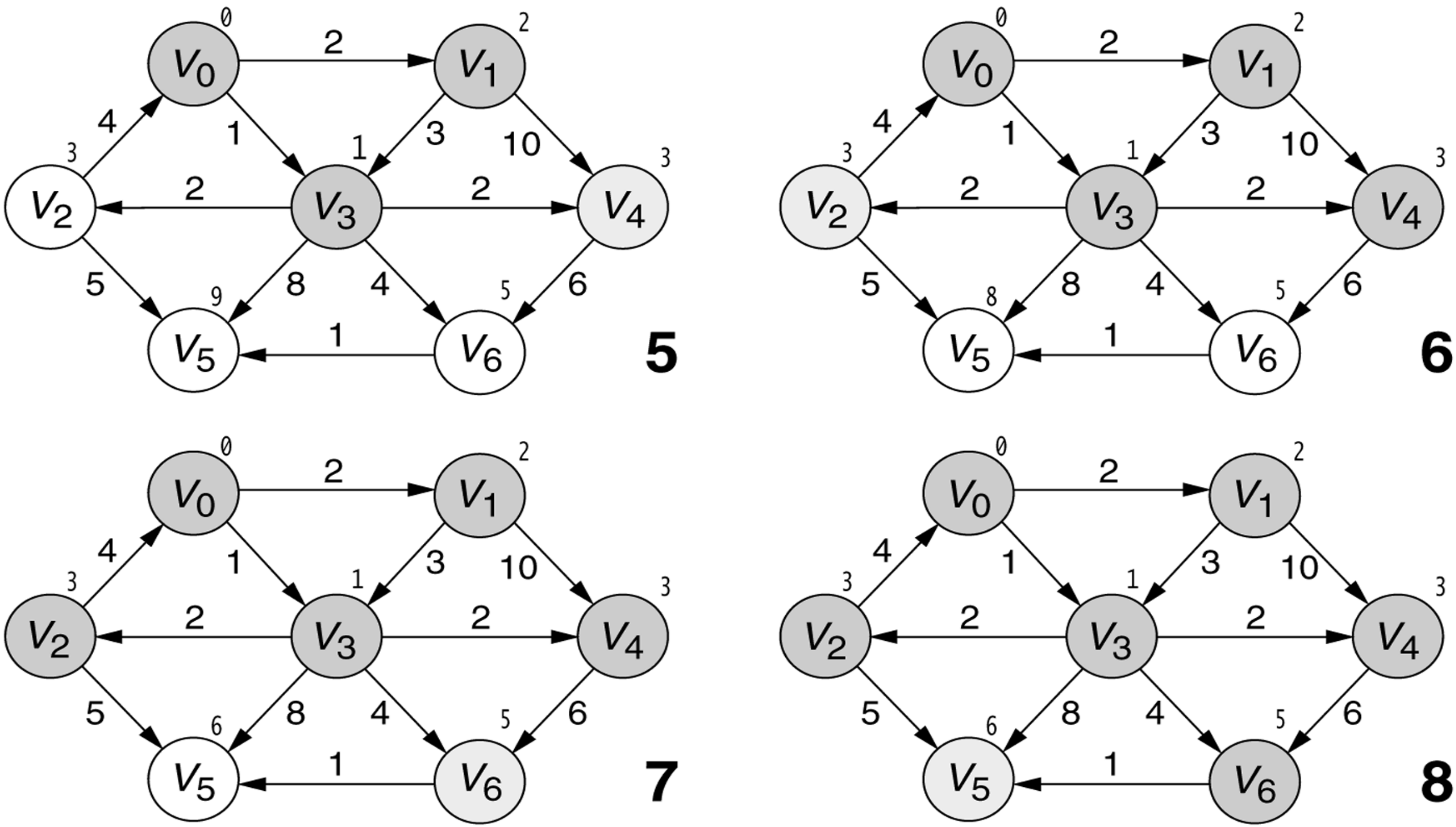


Figure 14.25B

Stages of Dijkstra's algorithm. The conventions are the same as those in Figure 14.21.



Implementation

- A queue is no longer appropriate for storing vertices to be visited.
- The priority queue is an appropriate data structure.
- Add a new entry consisting of a vertex and a distance, to the priority queue every time a vertex has its distance lowered.