Range Queries

(bonus: Sqrt Complexity and Computation)

CENG 213
METU/ODTÜ
Data Structures
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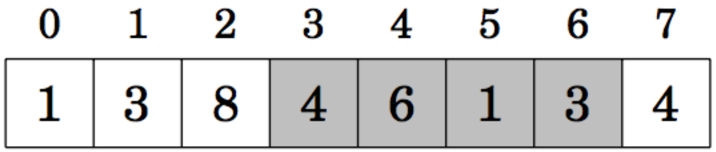
Goal

- Compute a value based on a subarray of an array.
- Consider range [3, 6] below.



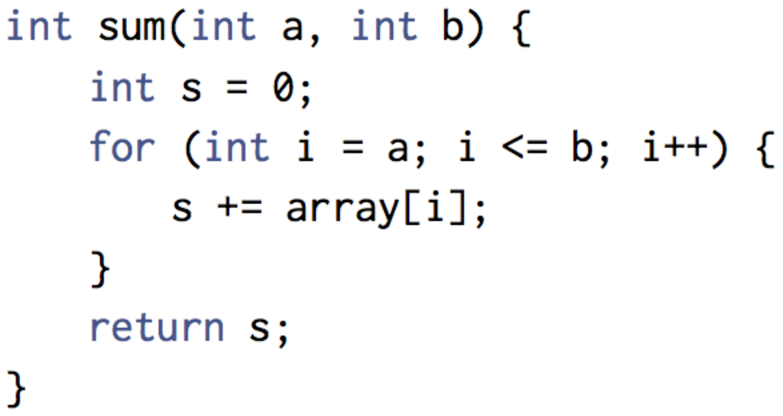
0	1	2	3	4	5	6	7
1	3	8	4	6	1	3	4

- $\text{sum}_q(3, 6) = 14$, $\text{min}_q(3, 6) = 1$, $\text{max}_q(3, 6) = 6$.

Goal

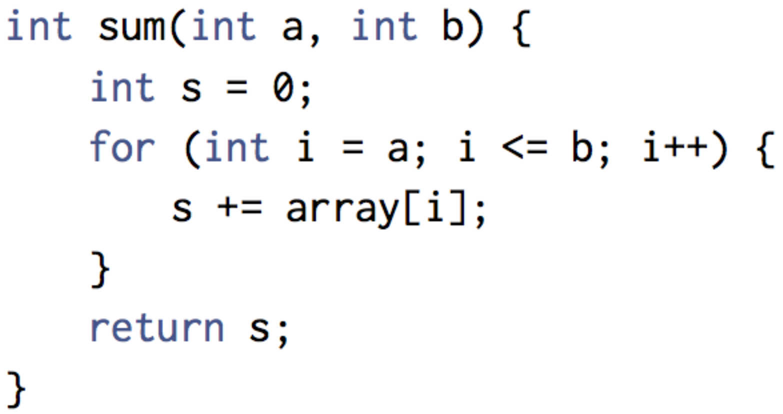
- Compute a value based on a subarray of an array.
- Typical range queries:
  - $\text{sum}_q(a,b)$: calculate the sum of values in range $[a,b]$.
  - $\text{min}_q(a,b)$: find the minimum value in range $[a,b]$.
  - $\text{max}_q(a,b)$: find the maximum value in range $[a,b]$.

Trivial Solution



```
int sum(int a, int b) {
    int s = 0;
    for (int i = a; i <= b; i++) {
        s += array[i];
    }
    return s;
}
```

Trivial Solution



```
int sum(int a, int b) {
    int s = 0;
    for (int i = a; i <= b; i++) {
        s += array[i];
    }
    return s;
}
```

- Works in $O(n)$ time, where n is the array size.
- We will make this fast!

Static Array Queries

- Assume array is static: values never updated.
- We will handle sum queries and min/max queries in this setting.

Prefix Sum Array

- Value at position k is $\text{sum}_q(0, k)$.
- Can be constructed in $O(n)$ time. How?

Array:

0	1	2	3	4	5	6	7
1	3	4	8	6	1	4	2

Prefix Sum:

0	1	2	3	4	5	6	7
1	4	8	16	22	23	27	29

Prefix Sum Array

- Value at position k is $\mathrm{sum}_q(0, k)$.
- Can be constructed in $O(n)$ time. How?
  - Dead simple application of dynamic programming.
    - `P[0]=A[0]; for(i=1 to n-1) P[i]=P[i-1]+A[i];`

Array:

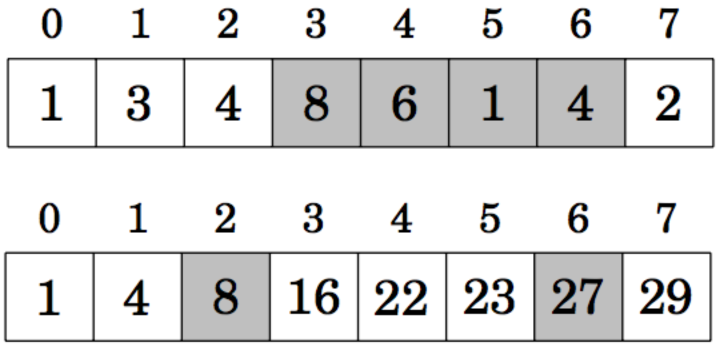
0	1	2	3	4	5	6	7
1	3	4	8	6	1	4	2

Prefix Sum:

0	1	2	3	4	5	6	7
1	4	8	16	22	23	27	29

Prefix Sum Array

- $\text{sum}_q(a,b) = \text{sum}_q(0,b) - \text{sum}_q(0,a-1)$
  - Define $\text{sum}_q(0,-1) = 0$.
- $O(n)$: $\text{sum}_q(3,6) = 8 + 6 + 1 + 4 = 19$.
- $O(1)$: $\text{sum}_q(3,6) = \text{sum}_q(0,6)$ - $\text{sum}_q(0,2) = 27 - 8$.

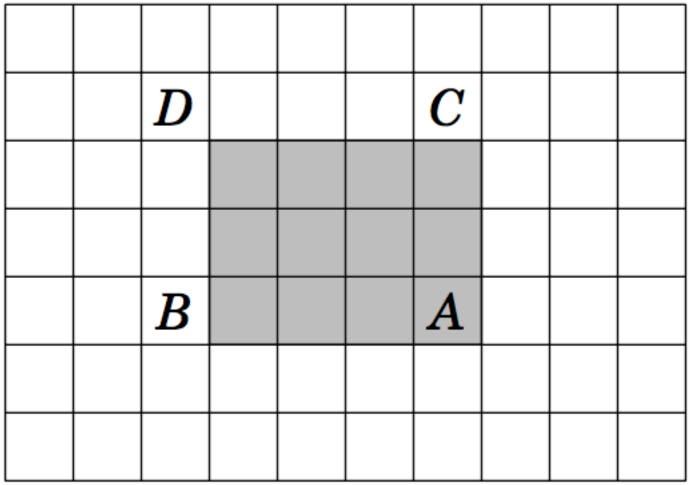


0	1	2	3	4	5	6	7
1	3	4	8	6	1	4	2

0	1	2	3	4	5	6	7
1	4	8	16	22	23	27	29

Prefix Sum Array

- Can be generalized to higher dimensions.



- Sum of gray subarray: $S(A)$ - $S(B)$ - $S(C)$ + $S(D)$ where $S(X)$ is the sum of values in a rectangular subarray from the upperleft corner to the pos. of X.

Sparse Table

- Handles minimum (and similarly max) queries.
- $O(n\log n)$ preprocessing, then all queries in $O(1)$.

Sparse Table

- Precompute all values of $\min_q(a,b)$ where b - a +1 (the length of the range) is a power of 2.

0	1	2	3	4	5	6	7
1	3	4	8	6	1	4	2

a	b	$\min_q(a,b)$
0	0	1
1	1	3
2	2	4
3	3	8
4	4	6
5	5	1
6	6	4
7	7	2

a	b	$\min_q(a,b)$
0	1	1
1	2	3
2	3	4
3	4	6
4	5	1
5	6	1
6	7	2

a	b	$\min_q(a,b)$
0	3	1
1	4	3
2	5	1
3	6	1
4	7	1
0	7	1

Sparse Table

- Precompute all values of $\min_q(a,b)$ where b - a +1 (the length of the range) is a power of 2.
- How many precomputed values?

0	1	2	3	4	5	6	7
1	3	4	8	6	1	4	2

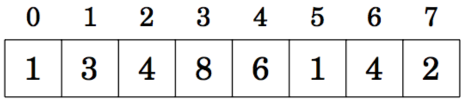
a	b	$\texttt{min}_q(a,b)$
0	0	1
1	1	3
2	2	4
3	3	8
4	4	6
5	5	1
6	6	4
7	7	2

a	b	$\texttt{min}_q(a,b)$
0	1	1
1	2	3
2	3	4
3	4	6
4	5	1
5	6	1
6	7	2

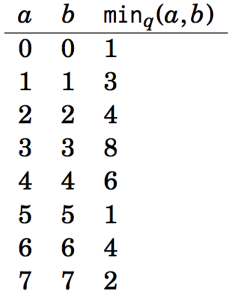
a	b	$\texttt{min}_q(a,b)$
0	3	1
1	4	3
2	5	1
3	6	1
4	7	1
0	7	1

Sparse Table

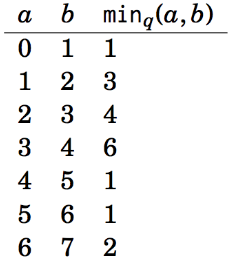
- Precompute all values of $\min_q(a,b)$ where b - a +1 (the length of the range) is a power of two.
- How many precomputed values?
  - $O(n\log n)$ because
    - there're $O(\log n)$ range lengths that are powers of 2.
    - there're $O(n)$ values at each range, e.g., n values for range of length 1, n-1 vals for range of length -1, ..



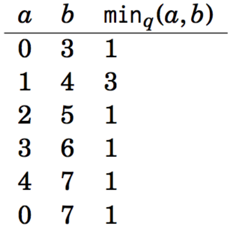
0	1	2	3	4	5	6	7
1	3	4	8	6	1	4	2



a	b	$\min_q(a,b)$
0	0	1
1	1	3
2	2	4
3	3	8
4	4	6
5	5	1
6	6	4
7	7	2



a	b	$\min_q(a,b)$
0	1	1
1	2	3
2	3	4
3	4	6
4	5	1
5	6	1
6	7	2



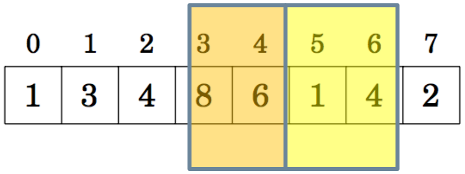
a	b	$\min_q(a,b)$
0	3	1
1	4	3
2	5	1
3	6	1
4	7	1
0	7	1

Sparse Table

- Precompute all values of $\min_q(a,b)$ where $b - a + 1$ (the length of the range) is a power of two.
- Each of the $O(n\log n)$ values will be computed in O(1) via the recursion (DP again!):

$$\min_q(a,b) = \mathbf{min}(\ \min_q(a,a+w-1), \min_q(a+w,b)\ )$$

where b-a+1 is a power of two and w = (b-a+1)/2 //mid.



0	1	2	3	4	5	6	7
1	3	4	8	6	1	4	2

a	b	$\min_q(a,b)$
0	0	1
1	1	3
2	2	4
3	3	8
4	4	6
5	5	1
6	6	4
7	7	2

a	b	$\min_q(a,b)$
0	1	1
1	2	3
2	3	4
3	4	6
4	5	1
5	6	1
6	7	2

a	b	$\min_q(a,b)$
0	3	1
1	4	3
2	5	1
3	6	1
4	7	1
0	7	1

Sparse Table

- Precompute all values of $\min_q(a,b)$ where b - a +1 (the length of the range) is a power of two.
- Each of the $O(n\log n)$ values will be computed in O(1) via the recursion (DP again!):
- Hence the $O(n\log n)$ preprocessing time.

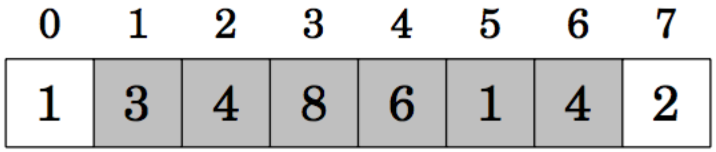
Sparse Table

- Query response in $O(1)$ via

$$\mathrm{min}_q(a,b) = \mathbf{min}(\ \mathrm{min}_q(a,a+k-1), \mathrm{min}_q(b-k+1,b)\ )$$

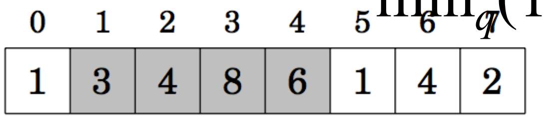
where k is the largest power of 2 that doesn't exceed b-a+1, the range length.

Here, the range $[a,b]$ is represented as the union of the ranges $[a,a+k-1]$ and $[b-k+1,b]$, both of length k.



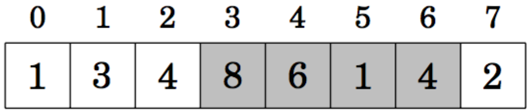
0	1	2	3	4	5	6	7
1	3	4	8	6	1	4	2

Range length 6, the largest power of 2 that doesn't exceed 6 is 4, k=4.



$\mathrm{min}_q(1,4)=3$

0	1	2	3	4	5	6	7
1	3	4	8	6	1	4	2



0	1	2	3	4	5	6	7
1	3	4	8	6	1	4	2

$\mathrm{min}_q(3,6)=1$

Dynamic Array Queries

- Now we will enable updates on array, hence dynamic.
- We will handle sum queries, min/max queries, and update queries in this setting.

Binary Indexed Tree*

- Dynamic variant of a Prefix Sum Array.
  - Handles range sum queries in $O(\log n)$ time. //PSA O(1)
  - Handles updating a values in $O(\log n)$ time. //PSA not⋆
  - Using two BITs make min queries possible.
    - This is more complex than using a Segment Tree (later).

⋆ PSA can handle this but needs $O(n)$ to rebuild PSA again.

* BIT aka Fenwick Tree.

Binary Indexed Tree

- Tree is conceptual; we actually maintain an array.
  - Array is 1-indexed to make the implementation easier.

Binary Indexed Tree

- Let $p(k)$ denote the largest pow of 2 that divides k. We store a BIT as an array such that

$$\text{tree}[k] = \text{sum}_q(k - p(k)+1,k)$$

- That is, each position k contains the sum of values in a range of the original array whose length is $p(k)$ and that ends at position k.
  - See slides 30-31 for the BIT construction.
- Since $p(6) = 2$, tree[6] contains value of $\text{sum}_q(5,6)$.

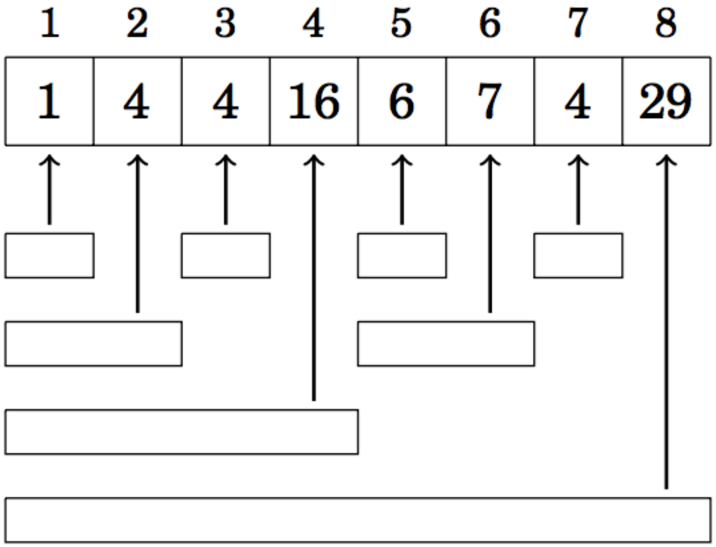
Binary Indexed Tree

- $\text{sum}_q(1,k)$ can be computed in $O(\log n)$ because a range $[1,k]$ can always be divided into $O(\log n)$ ranges whose sums are stored in the tree.

Array:

1	2	3	4	5	6	7	8
1	3	4	8	6	1	4	2

BIT:



1	2	3	4	5	6	7	8
1	4	4	16	6	7	4	29

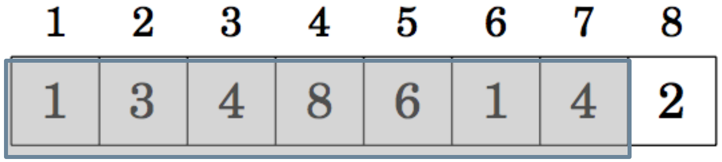
At most lg8=3 ranges to be used.

$[1,2^a]$ for the biggest $2^a < k$ solves half the problem, and so on w/ a--. ($2^a = k$ case solved in 1 shot.)

Binary Indexed Tree

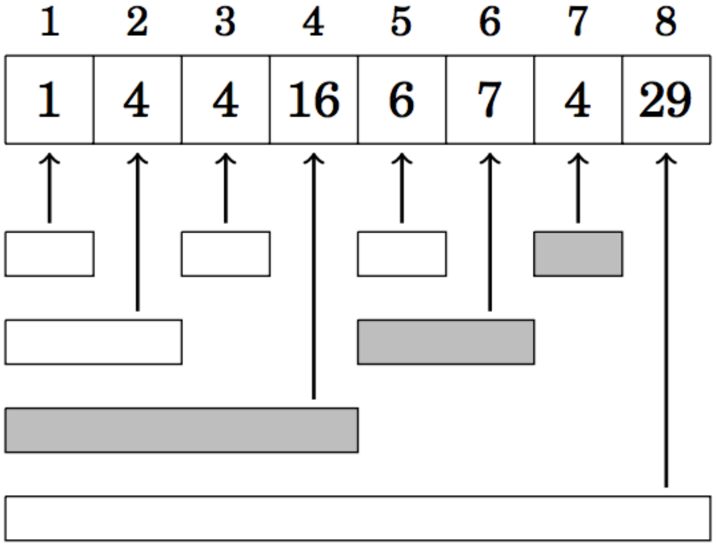
- $\text{sum}_q(1,k)$ can be computed in $O(\log n)$ because a range $[1,k]$ can always be divided into $O(\log n)$ ranges whose sums are stored in the tree.

Array:



1	2	3	4	5	6	7	8
1	3	4	8	6	1	4	2

BIT:



1	2	3	4	5	6	7	8
1	4	4	16	6	7	4	29

Binary Indexed Tree

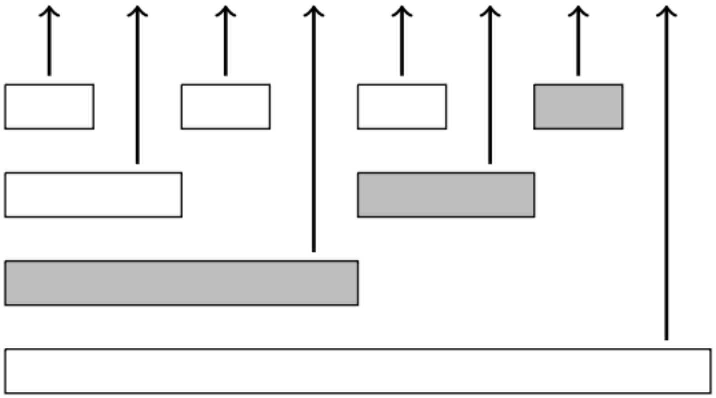
- $\text{sum}_q(1,7) = \text{sum}_q(1,4) + \text{sum}_q(5,6) + \text{sum}_q(7,7)$
  $= 16 + 7 + 4 = 27.$

Array:

1	2	3	4	5	6	7	8
1	3	4	8	6	1	4	2

BIT:

1	2	3	4	5	6	7	8
1	4	4	16	6	7	4	29



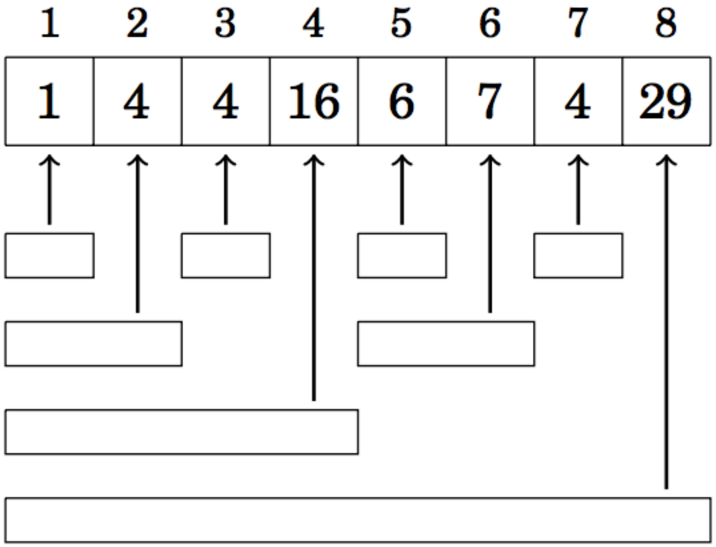
Binary Indexed Tree

- $\mathrm{sum}_q(a,b) = \mathrm{sum}_q(1,b) - \mathrm{sum}_q(1,a-1)$ //PSA trick for a>1
- $\mathrm{sum}_q(3,6) = \mathrm{sum}_q(1,6) - \mathrm{sum}_q(1,2) = 23 - 4 = 19.$

Array:

1	2	3	4	5	6	7	8
1	3	4	8	6	1	4	2

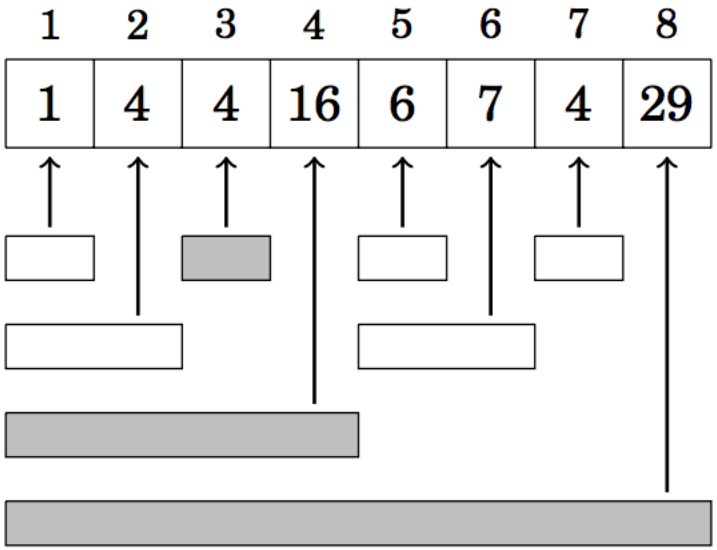
BIT:



1	2	3	4	5	6	7	8
1	4	4	16	6	7	4	29

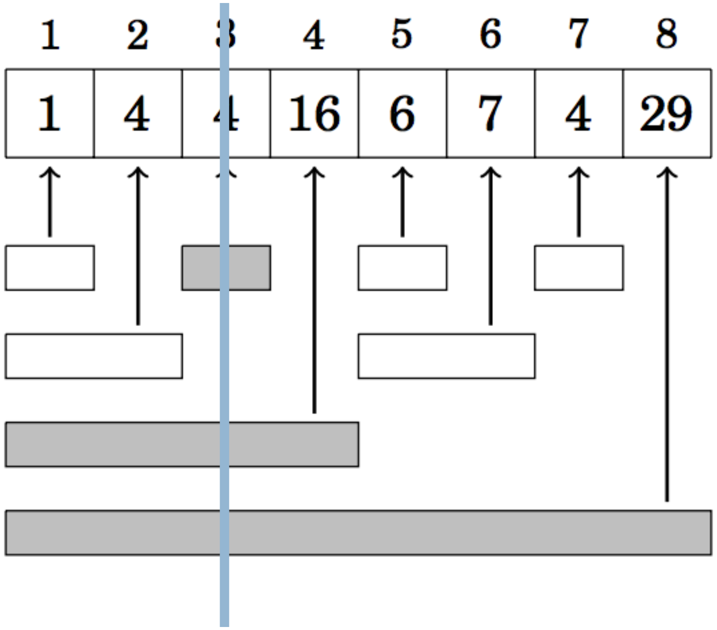
Binary Indexed Tree

- After updating a value in the array, several values in the BIT should be updated.
- If the value at position 3 changes, the sums of the following ranges change:



Binary Indexed Tree

- After updating a value in the array, several values in the BIT should be updated.
- Each array element belongs to $O(\log n)$ ranges, hence update cost is $O(\log n)$.



At most lg8=3 ranges to be updated.

Last bar is definetely involved, then either left or right side of it, and so on.

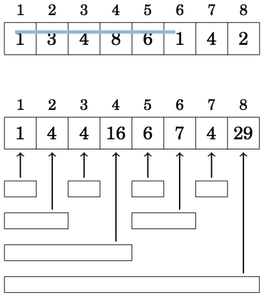
Binary Indexed Tree

- Implementation made efficient via bit operations.

$p(k) = k \;\&\; \text{-}k$ //largest pow of 2 that divides k.
//zeroes all the bits except the last set one.
//$p(6)$=2: 0110 → 0010, $p(7)$=1: 0111 → 0001, ..

- Computation of $\text{sum}_q(1,k)$:
- $O(\log n)$ values are accessed and each move to the next position takes $O(1)$ time.

```
int sum(int k) {
    int s = 0;
    while (k >= 1) {
        s += tree[k];
        k -= k&-k;
    }
    return s;
}
```



- Since $p(6) = 2$, tree[6] contains value of $\text{sum}_q(5,6)$ //length of range is 2.
- Since $p(4) = 4$, tree[4] contains value of $\text{sum}_q(1,4)$ //length of range is 4.

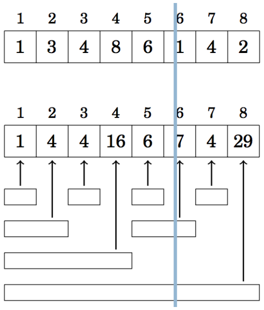
Binary Indexed Tree

- Implementation made efficient via bit operations.

$$p(k) = k \,\&\, \text{-}k \quad \text{//largest pow of 2 that divides } k$$

- Addition of x to position k:
- $O(\log n)$ values are accessed and each move to the next position takes $O(1)$ time.

```
void add(int k, int x) {
    while (k <= n) {
        tree[k] += x;
        k += k&-k;
    }
}
```



1	2	3	4	5	6	7	8
1	3	4	8	6	1	4	2

1	2	3	4	5	6	7	8
1	4	4	16	6	7	4	29

- Since $p(6) = 2$, tree[6] contains value of $\text{sum}_q(5,6)$ //length of range is 2.
- Since $p(8) = 8$, tree[8] contains value of $\text{sum}_q(1,8)$ //length of range is 8.

Binary Indexed Tree

- Implementation made efficient via bit operations.

$$p(k) = k \,\&\, \text{-}k \quad \text{//largest pow of 2 that divides } k$$

- Initial construction of a BIT is $O(n\log n)$.
  - Initialize all elements to 0.
  - Fill all range sums (of length $p(k)$).
    - Call add() n times using the input values: add(1..n,A[i]).

Binary Indexed Tree

- Implementation made efficient via bit operations.

$$p(k) = k \;\&\; \text{-}k \quad \text{//largest pow of 2 that divides } k$$

- Initial construction of a BIT is $O(n)$.
  - Construct a PSA in $O(n)$.
  - Fill all range sums (of length $p(k)$).
    - Use PSA lookups in $O(1)$ time per sum.

Segment Tree

- A more general data structure than BIT.
  - BIT supports sum queries (min queries possible but complicated).
  - ST supports sum, min, max, gcd, xor in $O(\log n)$ time.
  - ST takes more memory and is harder to implement.

Segment Tree

- Tree is conceptual; we actually maintain an array.
  - Array is 0-indexd* to make the implementation easier.
  - Array size is a power of 2 to make the implmtn easier.
    - Append extra elements to get this property, if necessary.

* Query ranges are 0-based but the tree array 1-based.

(will be clear in Slide 41)

Segment Tree

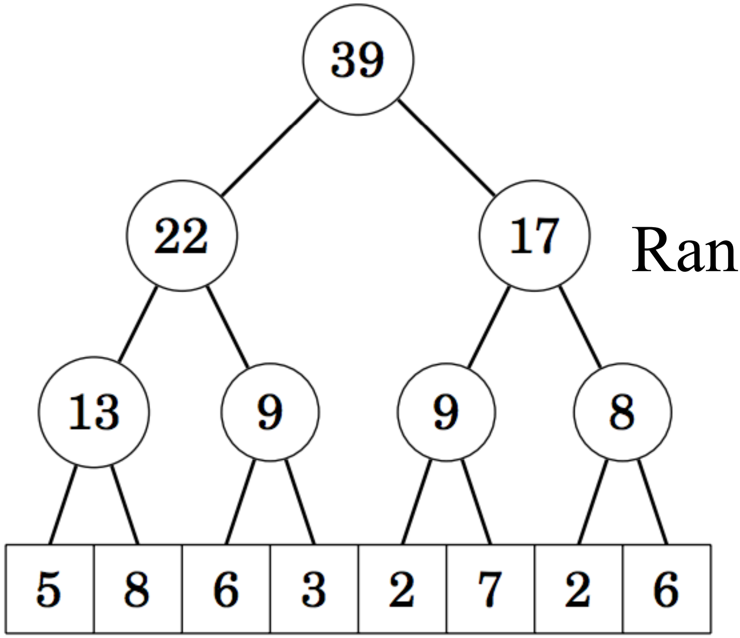
- Each internal tree node stores a value based on an array range whose length is a power of 2.

Array:

0	1	2	3	4	5	6	7
5	8	6	3	2	7	2	6

goes to leaves.

ST (sum_q):



Range length is $2^2 = 4$.

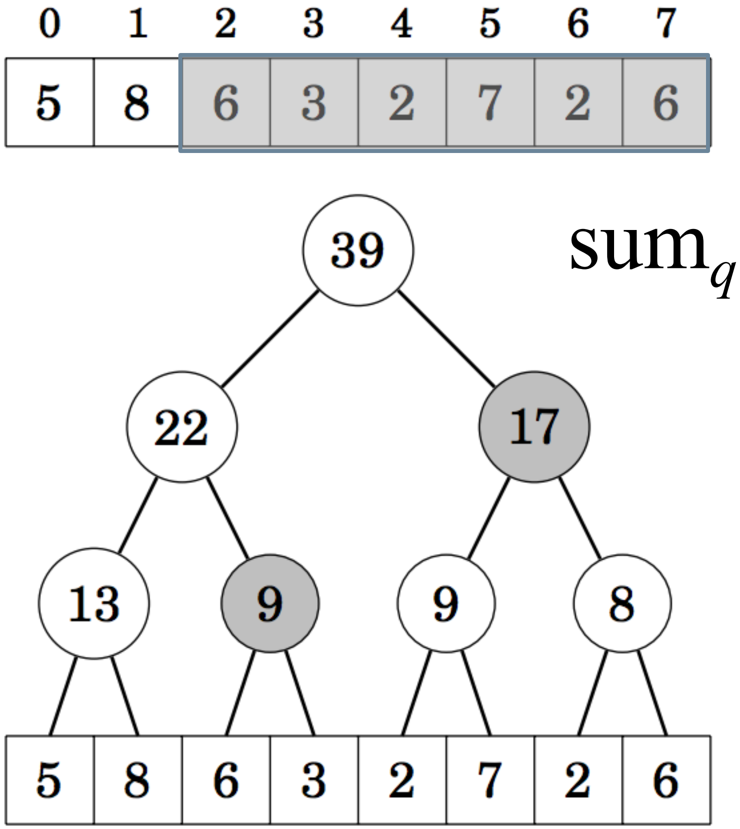
Segment Tree

- Any range [*a*,*b*] can be divided into *O*(log*n*) ranges whose values are stored in tree nodes.

Array:

ST (sum_q):

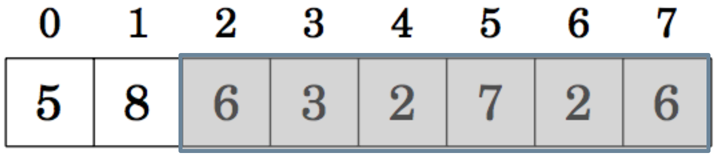
$\text{sum}_q(2,7) = 9 + 17$



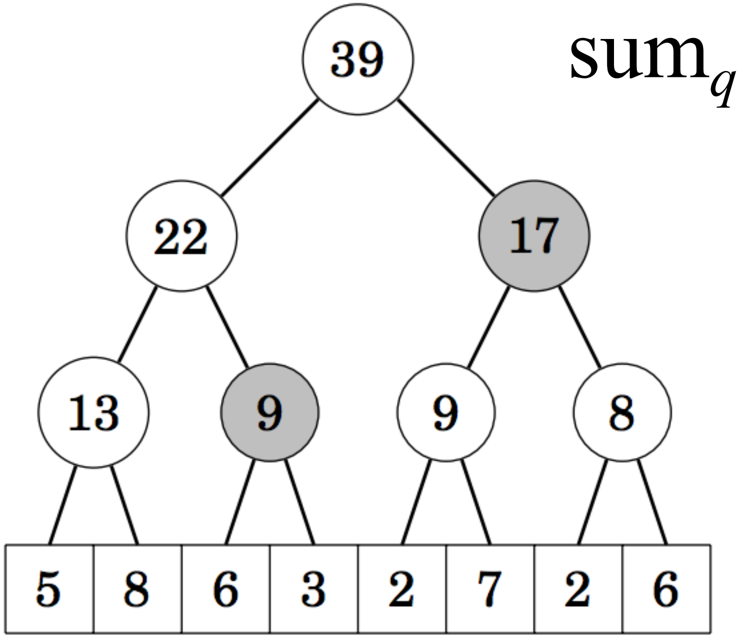
Segment Tree

- At most 2 nodes on each level needed ➔ $O(\log n)$ nodes/ranges needed, so sum_q complexity is $O(\log n)$.

Array:



ST (sum_q):



$\text{sum}_q(2,7) = 9 + 17$

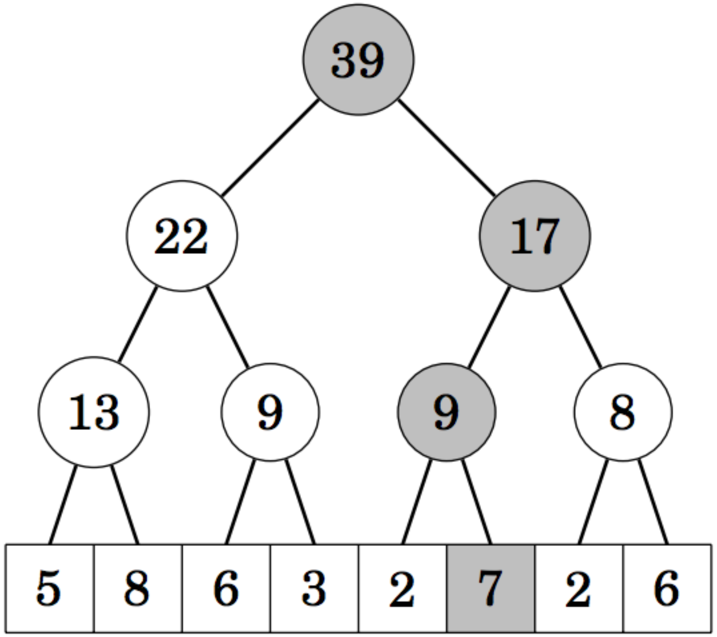
Segment Tree

- After an update, update all nodes whose value depends on the updated value.

Array:

0	1	2	3	4	5	6	7
5	8	6	3	2	7	2	6

ST (sum_q):



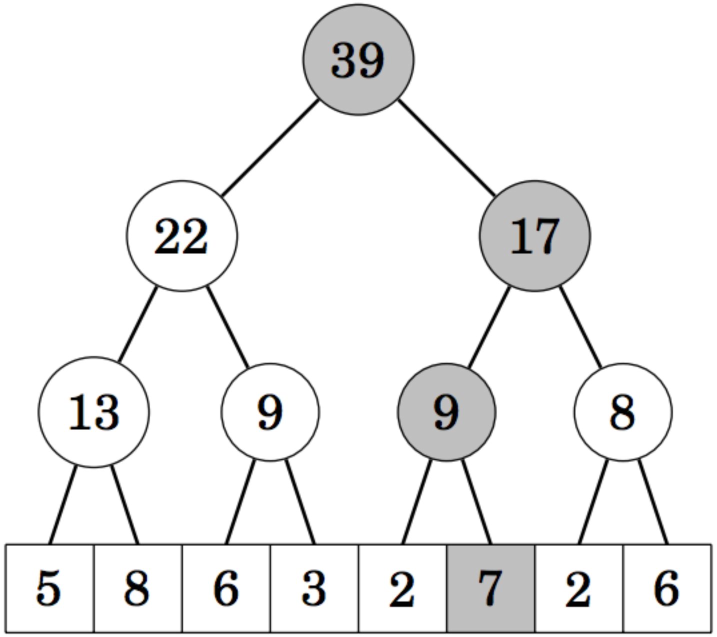
Segment Tree

- Do this by traversing the path from the updated element to root and updating nodes along the path.

Array:

0	1	2	3	4	5	6	7
5	8	6	3	2	7	2	6

ST (sum_q):

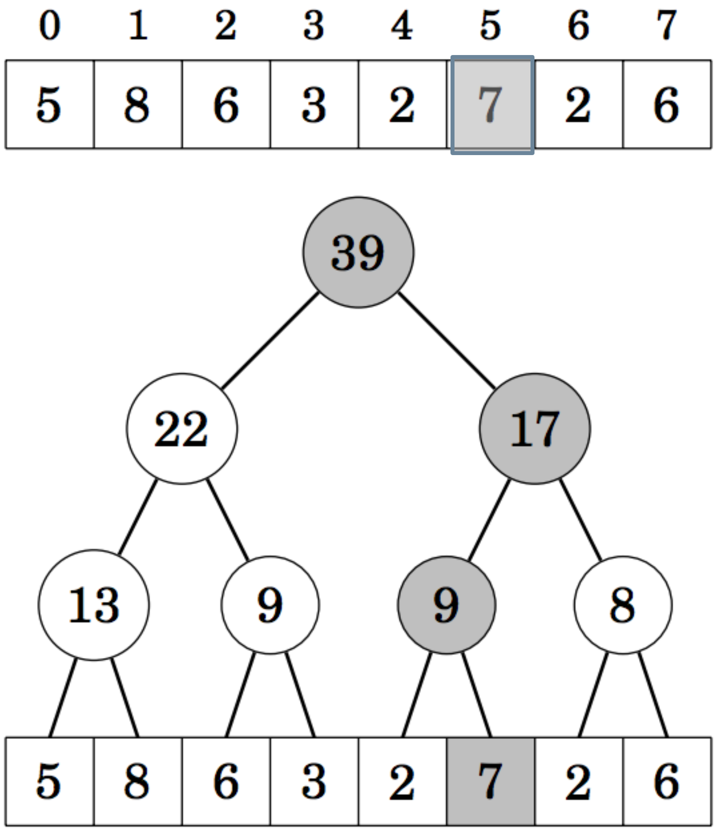


Segment Tree

- The path from bottom to top always consists of $O(\log n)$ nodes, so update complexity is $O(\log n)$.

Array:

ST (sum_q):

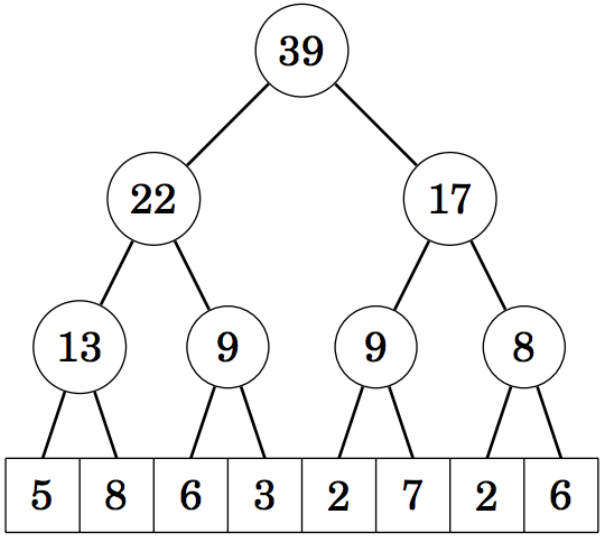


Segment Tree

- Implementation with an array of $2n$ elements where n is the size of the original array and a power of 2.

Segment Tree

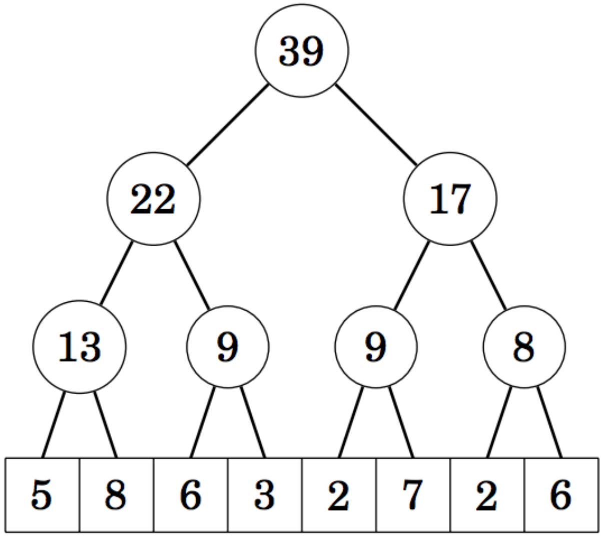
- Tree nodes stored from top to bottom.
  - tree[1] is the root, tree[2] and tree[3] its children, ..
  - tree[n] to tree[$2n$-1], the bottom level, input values.



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
39	22	17	13	9	9	8	5	8	6	3	2	7	2	6

Segment Tree

- Parent of tree[k] is tree[$\lfloor k/2 \rfloor$].
- Children of tree[k] is tree[$2k$] and tree[$2k+1$].

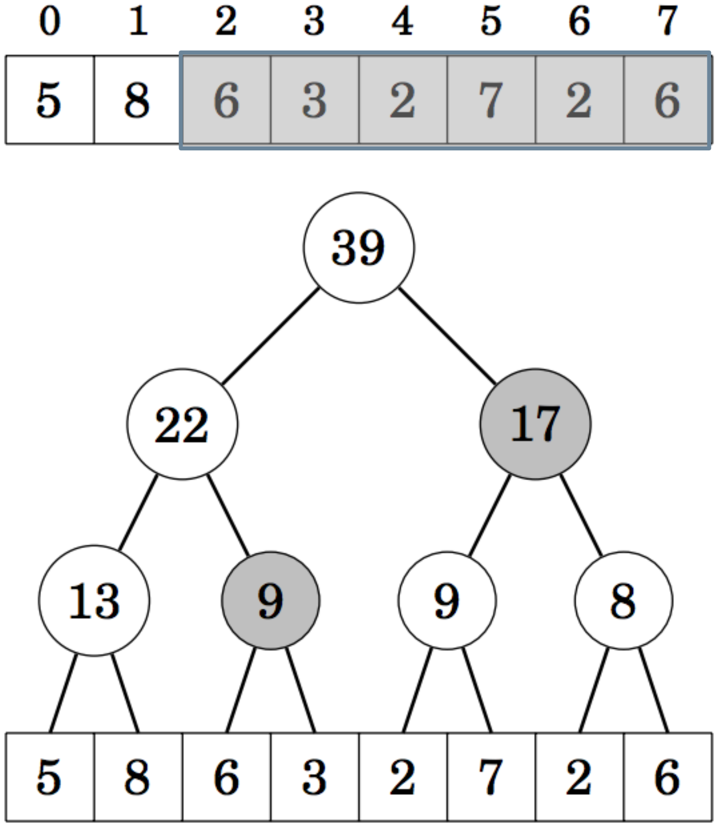


1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
39	22	17	13	9	9	8	5	8	6	3	2	7	2	6

Segment Tree

- sum$_q$(a,b) in $O(\log n)$ because ST has $O(\log n)$ levels and we move one level higher at each step.

```
int sum(int a, int b) {
    a += n; b += n; //range initially [a+n,b+n].
    int s = 0;
    while (a <= b) {
        if (a%2 == 1) s += tree[a++];
        if (b%2 == 0) s += tree[b--];
        a /= 2; b /= 2;
    }
    return s;
}
```

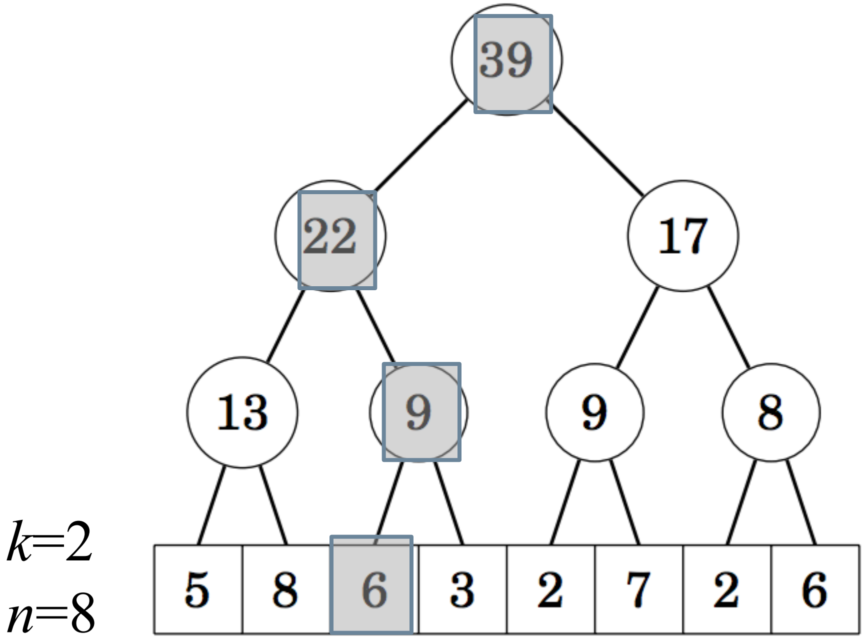


sum$_q$(2,7) = 9 + 17

Segment Tree

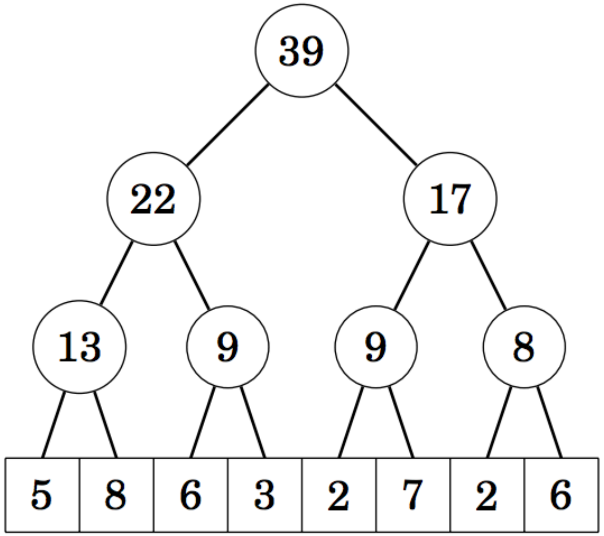
- add() increases the array value at position *k* by *x* in *O*(log*n*) because ST has *O*(log*n*) levels and we move one level higher at each step.

```
void add(int k, int x) {
    k += n;
    tree[k] += x;
    for (k /= 2; k >= 1; k /= 2) {
        tree[k] = tree[2*k]+tree[2*k+1];
    }
}
```



Segment Tree

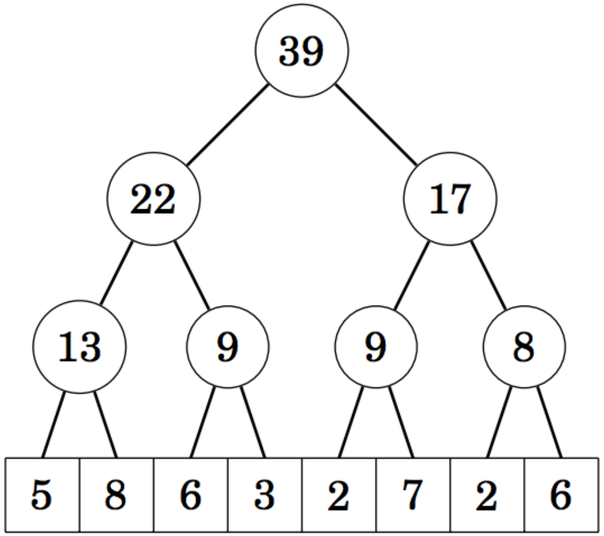
- ST can be constructed in $O(n)$. How?



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
39	22	17	13	9	9	8	5	8	6	3	2	7	2	6

Segment Tree

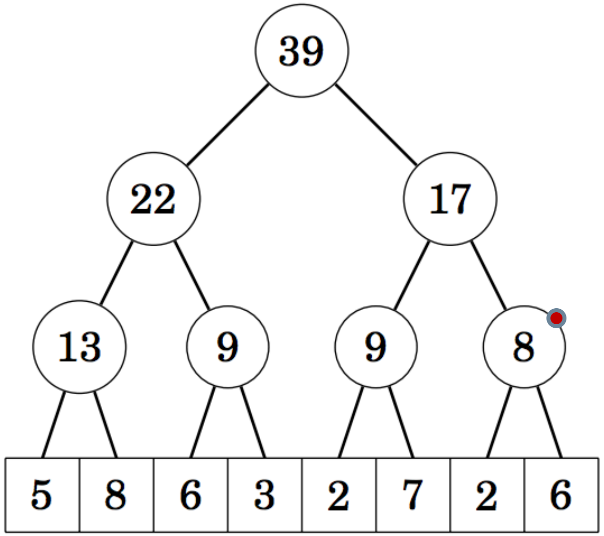
- ST can be constructed in $O(n)$. How?
  - Calling add n times on initially 0 array is not $O(n)$; it'd be $O(n\log n)$.



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
39	22	17	13	9	9	8	5	8	6	3	2	7	2	6

Segment Tree

- Go from the last intermediate node to the first (root), fill their values by adding their children at indices 2*k* and 2*k*+1. Each visited once, hence *O*(*n*).



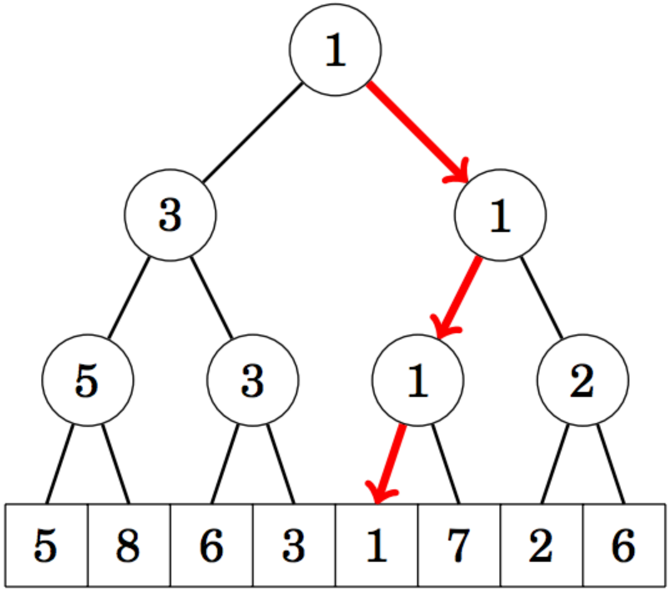
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
39	22	17	13	9	9	8	5	8	6	3	2	7	2	6

Segment Tree

- ST can also be used for min queries.
- Divide a range into two parts, compute the answer separately for both parts and then combine answers.
- Already did this for the sum queries.
- Similarly, it handles max, gcd, bit op (xor) queries.

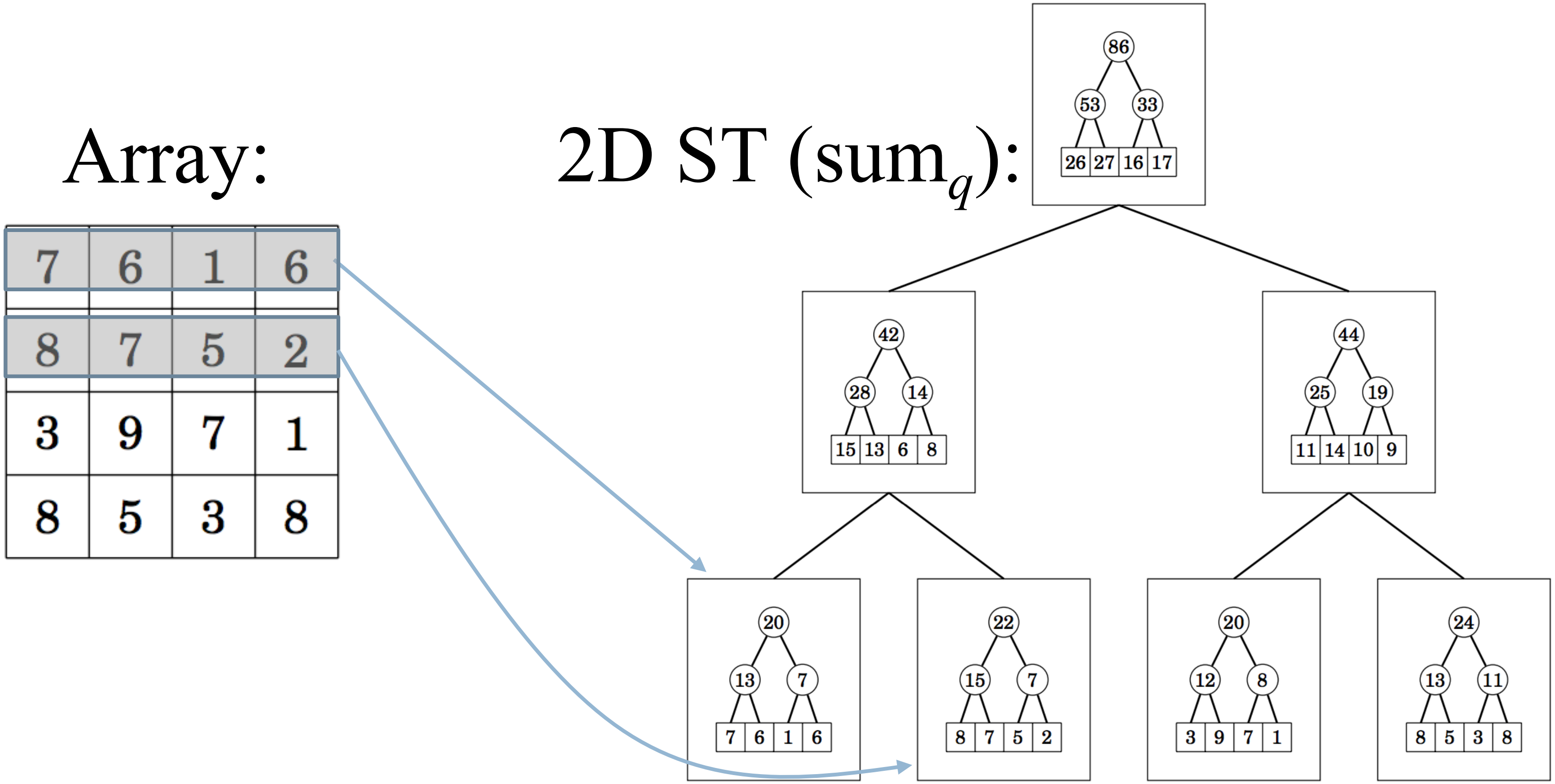
Segment Tree

- ST can also be used for min queries.
- Every tree node contains the smallest value in the corresponding array range.
- Instead of sums, minima are computed.



2D Segment Tree

- Segment Tree of Segment Trees.
- Supports rectangular subarray queries to a 2D array.



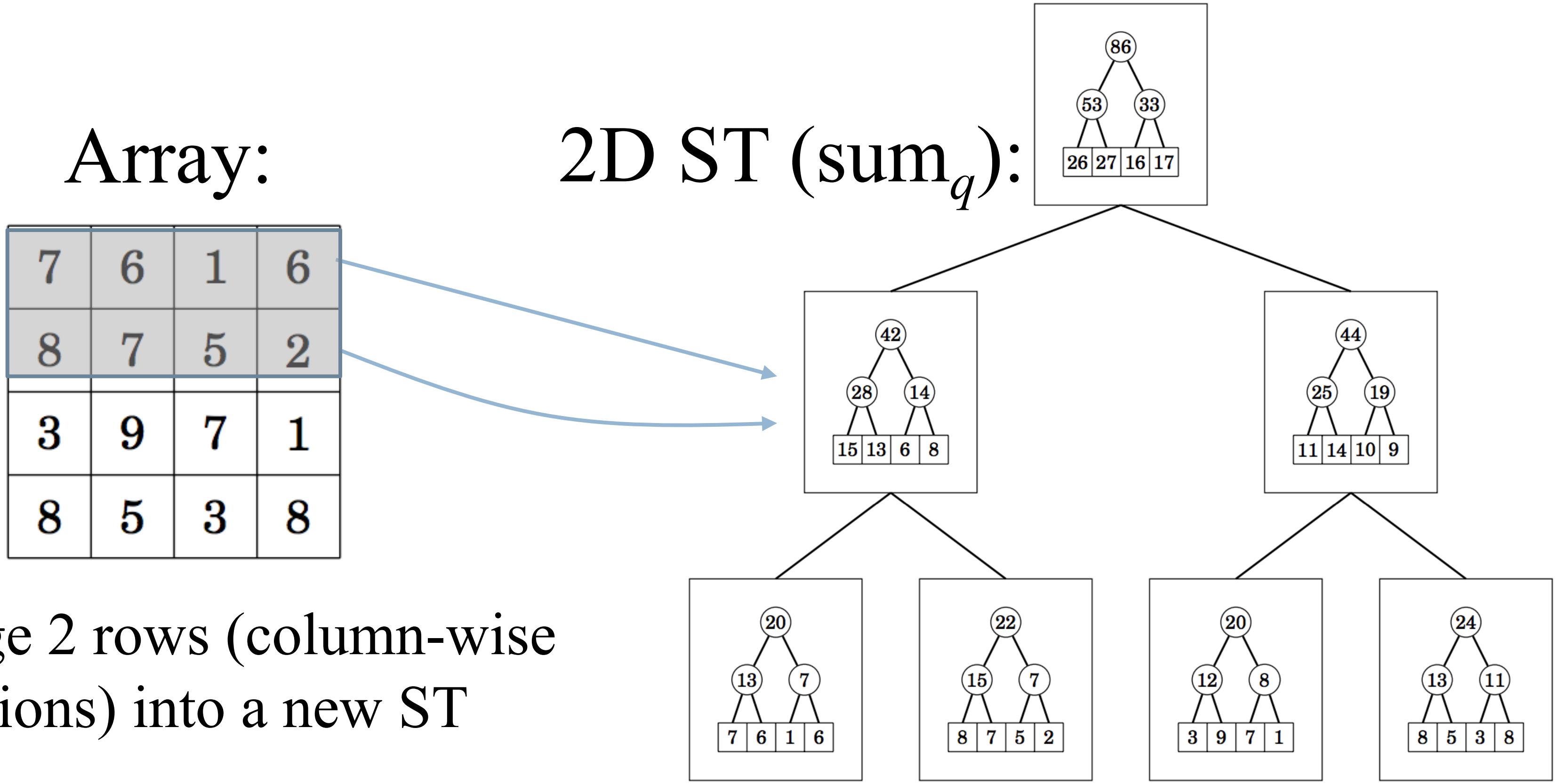
Array:

7	6	1	6
8	7	5	2
3	9	7	1
8	5	3	8

2D ST (sum_q):

2D Segment Tree

- Segment Tree of Segment Trees.
- Supports rectangular subarray queries to a 2D array.



Array:

7	6	1	6
8	7	5	2
3	9	7	1
8	5	3	8

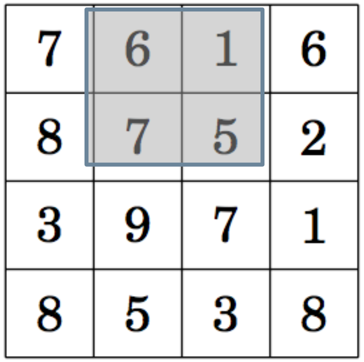
2D ST (sum_q):

Merge 2 rows (column-wise additions) into a new ST

2D Segment Tree

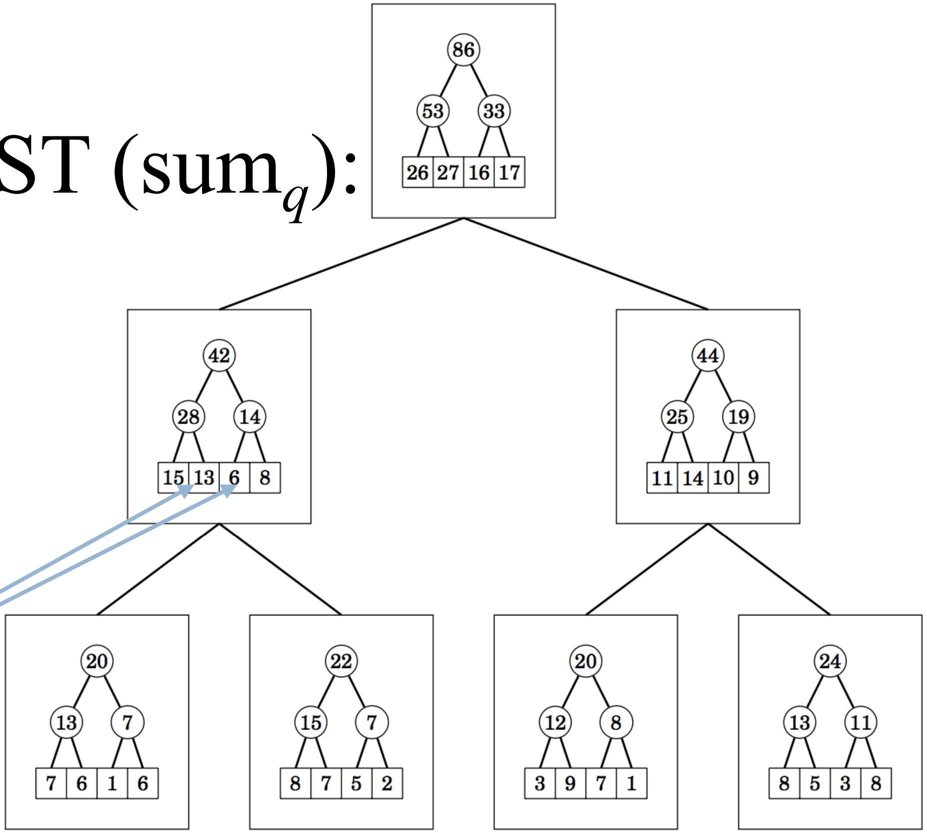
- Segment Tree of Segment Trees.
- Supports rectangular subarray queries to a 2D array.

Array:



7	6	1	6
8	7	5	2
3	9	7	1
8	5	3	8

2D ST (sum_q):



Sum for gray region can be obtained from the merged ranges

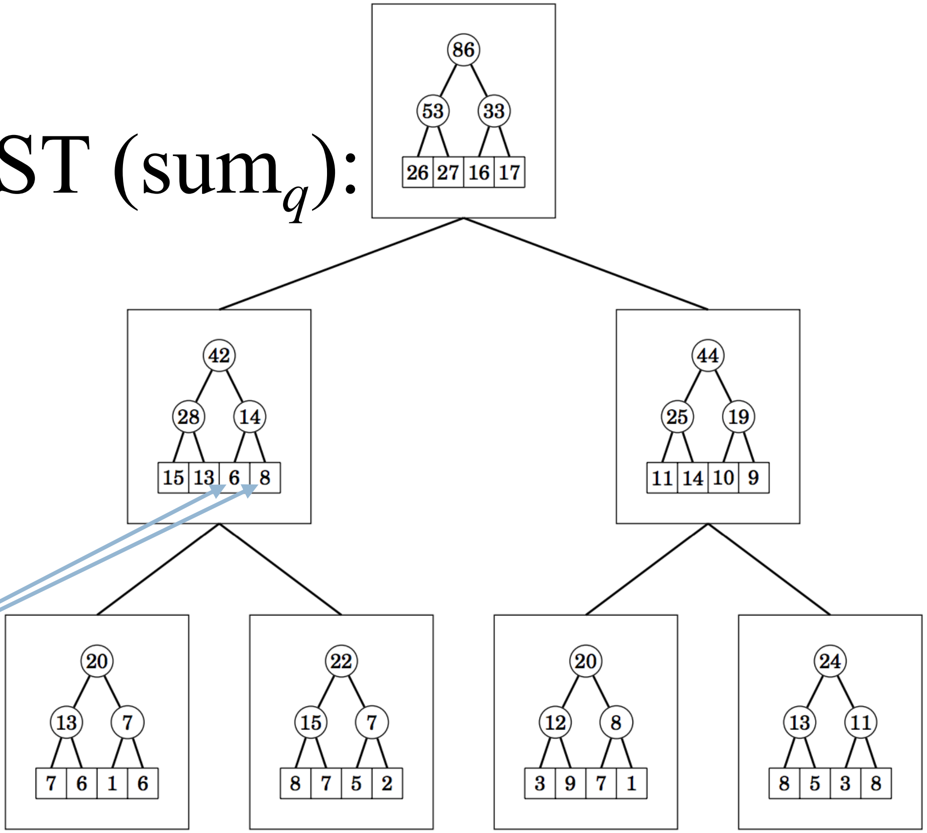
2D Segment Tree

- Segment Tree of Segment Trees.
- Supports rectangular subarray queries to a 2D array.

Array:

7	6	1	6
8	7	5	2
3	9	7	1
8	5	3	8

2D ST (sum_q):



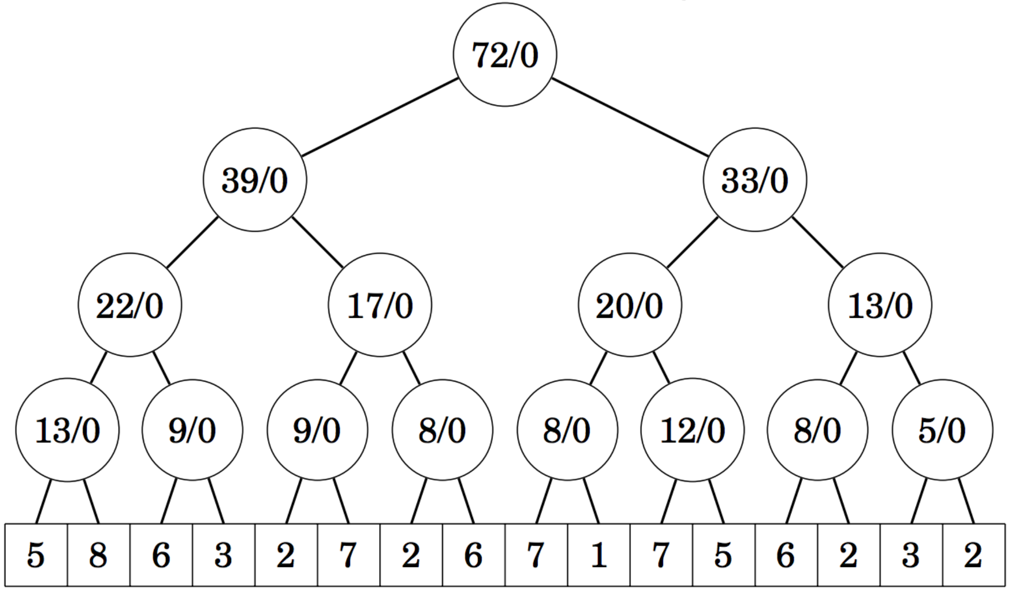
Sum for gray region can be obtained from the merged ranges

Lazy Propagation

- An optimization to make range updates faster.
- When there are many updates and updates are done on a range, we can postpone some updates and do those updates only when required.

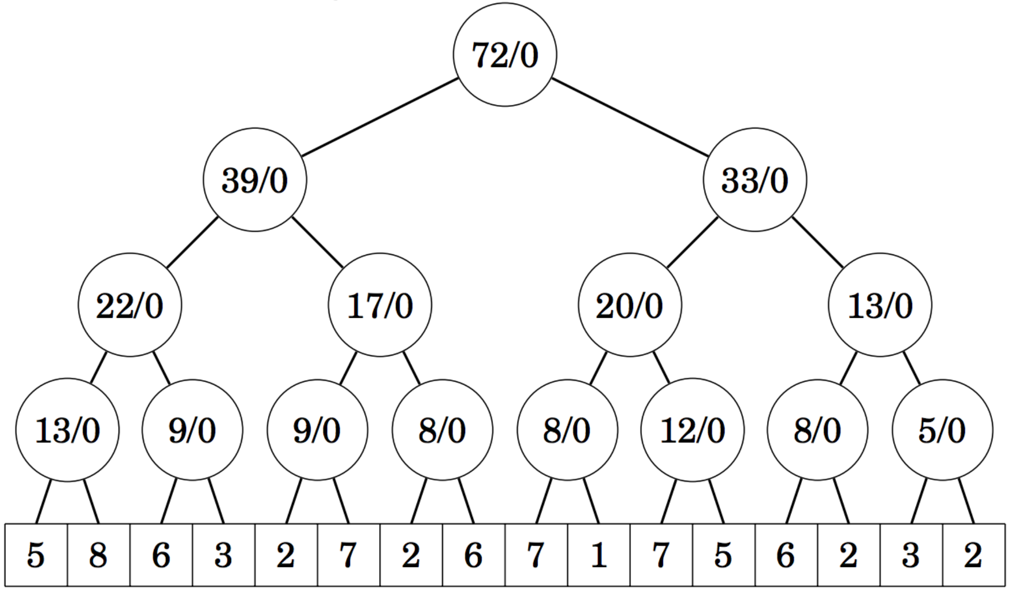
Lazy Propagation

- *s*/*z*: sum of values in the range / value of a lazy update.



Lazy Propagation

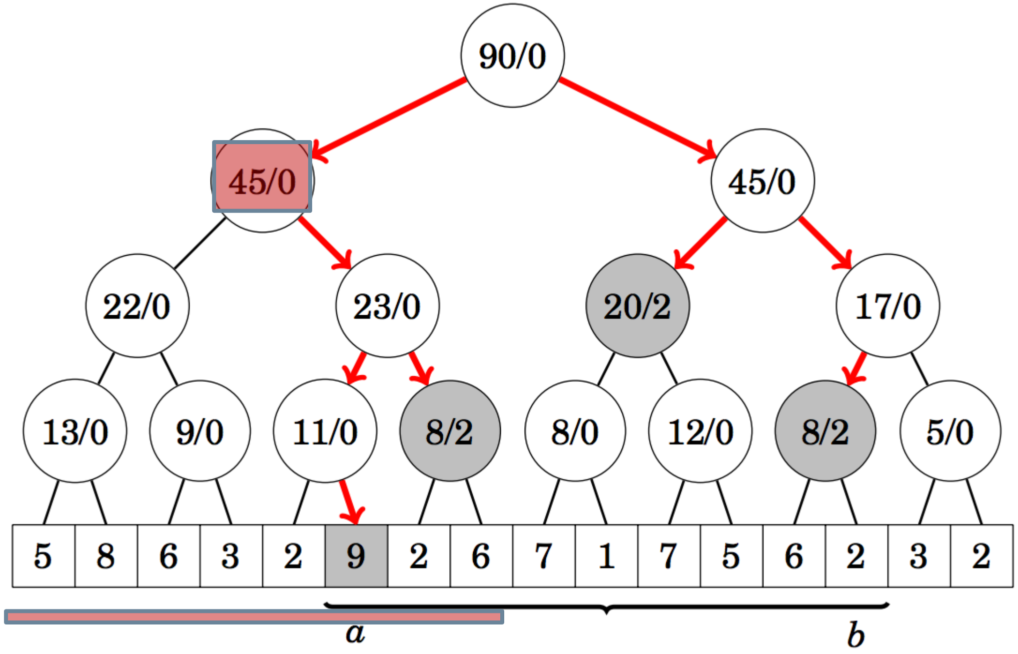
- ST after increasing *all* the elements in [a,b] by 2.



- When the elements in [a,b] are increased by u, we walk from the root towards the leaves and modify the nodes of the tree as follows.

Lazy Propagation

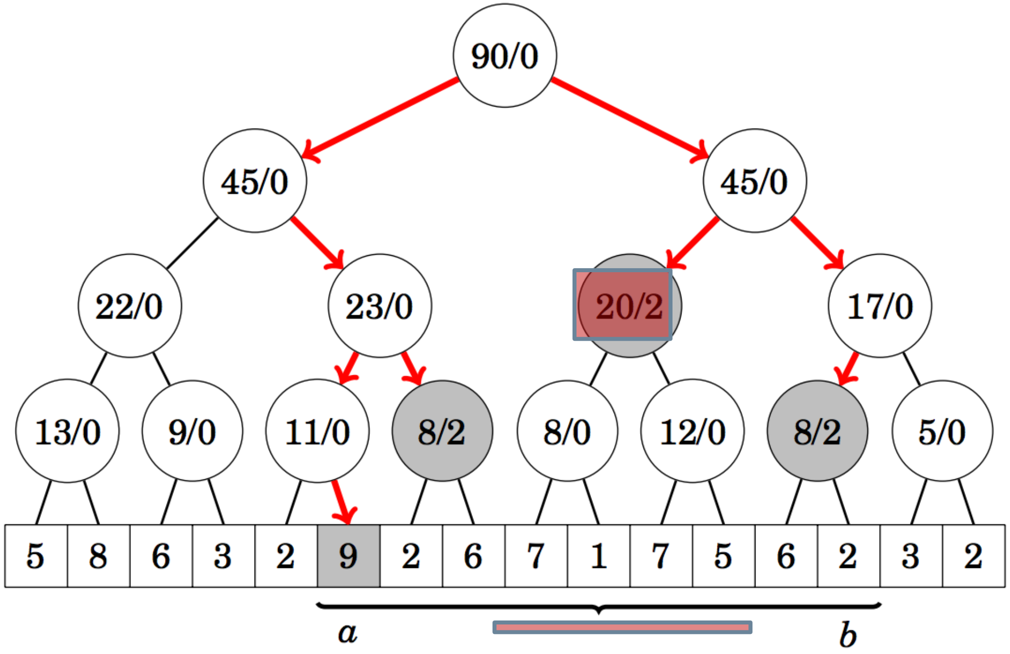
- ST after increasing *all* the elements in [a,b] by 2.



- If [x,y] is partially inside [a,b], we increase the s value of the node by hu, where h is the size of the intersection of [a,b] and [x,y], and recur.

Lazy Propagation

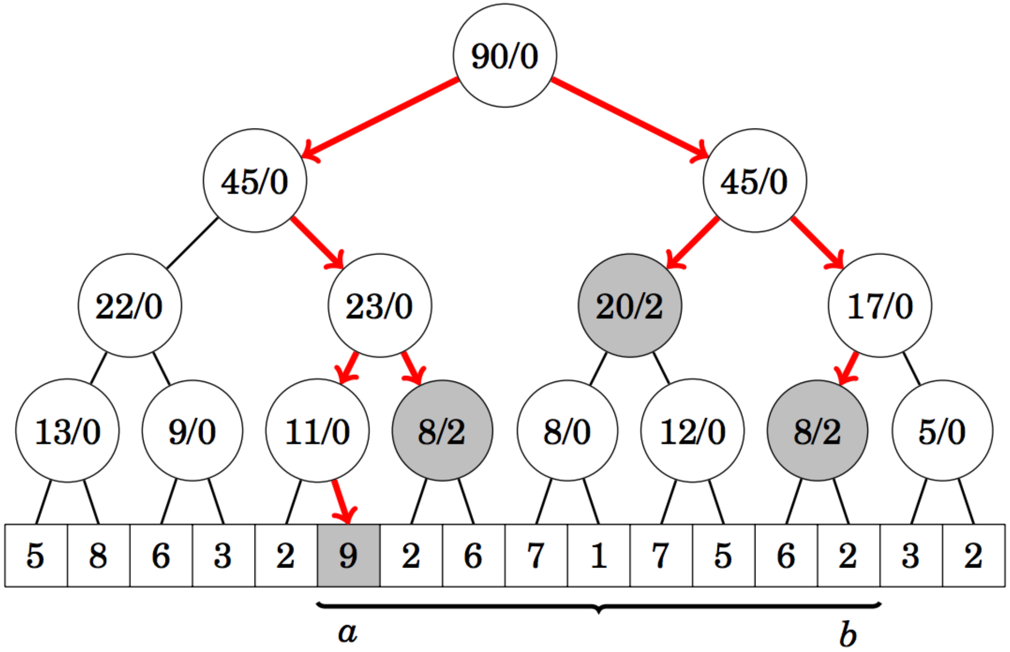
- ST after increasing *all* the elements in [*a*,*b*] by 2.



- If [*x*,*y*] is completely inside [*a*,*b*], we increase the *z* value of the node by *u*, and stop.

Lazy Propagation

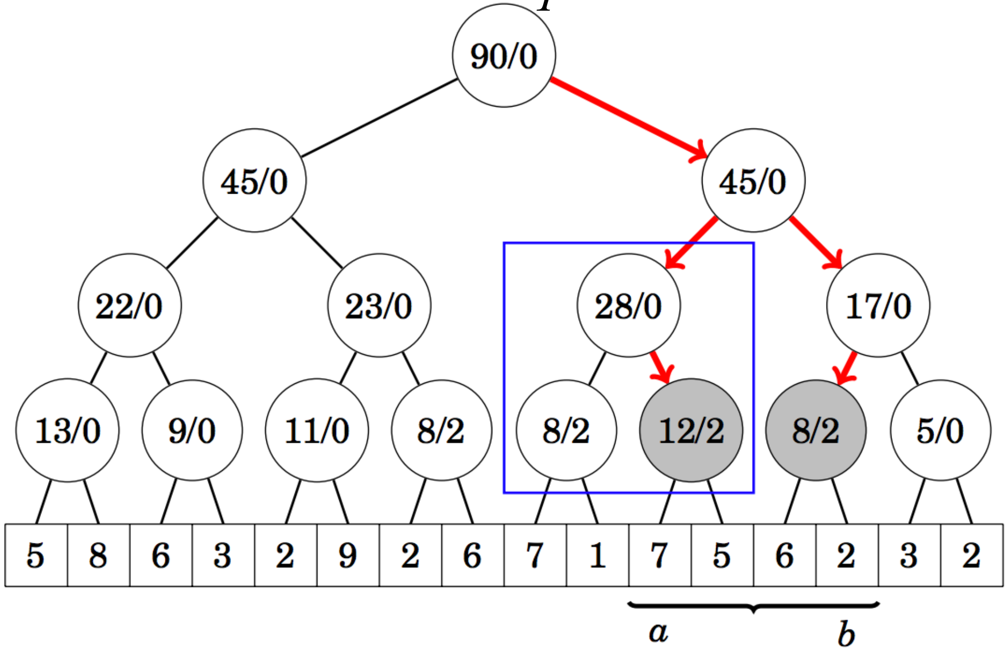
- ST after increasing *all* the elements in [a,b] by 2.



- The idea is that updates will be propagated downwards only when it is necessary, which guarantees that the operations are always efficient.

Lazy Propagation

- ST after computing $\text{sum}_q(a,b)$.



- Notice how the lazy update is applied to 28, and propagated below to 8 and 2 (blue part).

Additional Technique

- Increasing *all* the elements in [a,b] by x can also be done via Difference Array – has nothing to do w/ ST.

Array:

0	1	2	3	4	5	6	7
3	3	1	1	1	5	2	2

DA:

0	1	2	3	4	5	6	7
3	0	−2	0	0	4	−3	0

- DA indicates the differences between consecutive values in the original array A.
- Thus, A is the prefix sum array of the DA.

Additional Technique

Array:

0	1	2	3	4	5	6	7
3	3	1	1	1	5	2	2

DA:

0	1	2	3	4	5	6	7
3	0	−2	0	0	4	−3	0

- We can update a range in A by changing just two elements in DA: to increase A[1,4] by 5, it suffices to increase DA[1] by 5 and decrease DA[5] by 5.

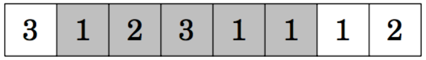
0	1	2	3	4	5	6	7
3	5	−2	0	0	−1	−3	0

- General, [*a*,*b*] by *x* → DA[*a*]+=*x* and DA[*b*+1]-=*x*, hence just 2 updates to update *O*(*n*)-range: *O*(1).

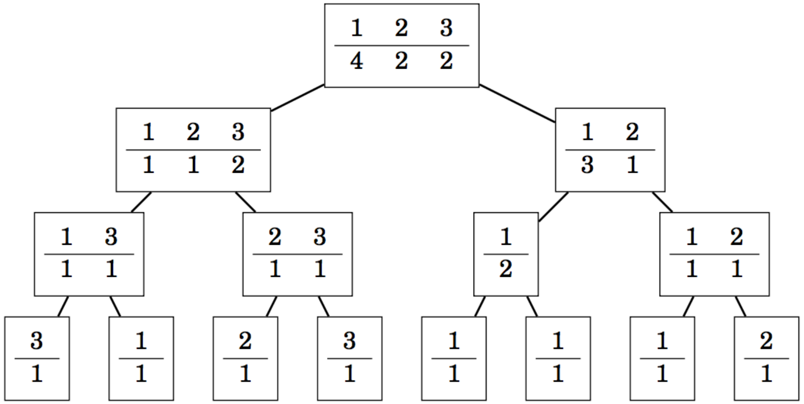
Segment Tree w/ DS Nodes

- Nodes contain data structures that maintain info about the corresponding ranges.
- ST supporting "how many times does x appear in the range $[a,b]$?".

Array:



ST:



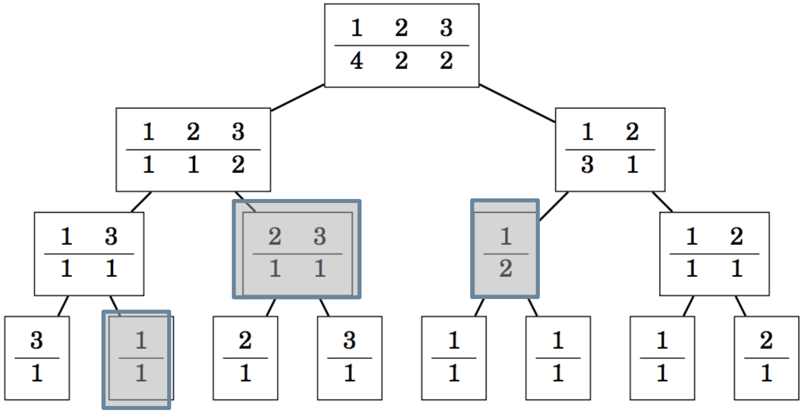
Segment Tree w/ DS Nodes

- Nodes contain data structures that maintain info about the corresponding ranges.
- ST supporting "how many times does x appear in the range $[a,b]$?".

Array:

3	1	2	3	1	1	1	2

ST:



- Query answered by combining results from nodes that belong to the range.

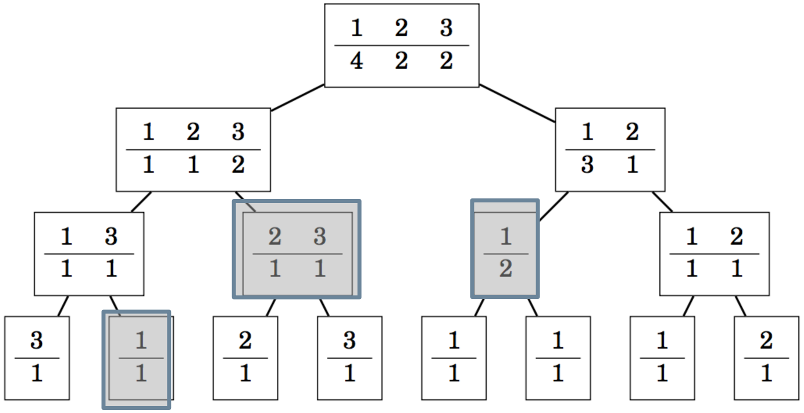
Segment Tree w/ DS Nodes

- Nodes contain data structures that maintain info about the corresponding ranges.
- ST supporting "how many times does x appear in the range $[a,b]$?".

Array:

3	1	2	3	1	1	1	2

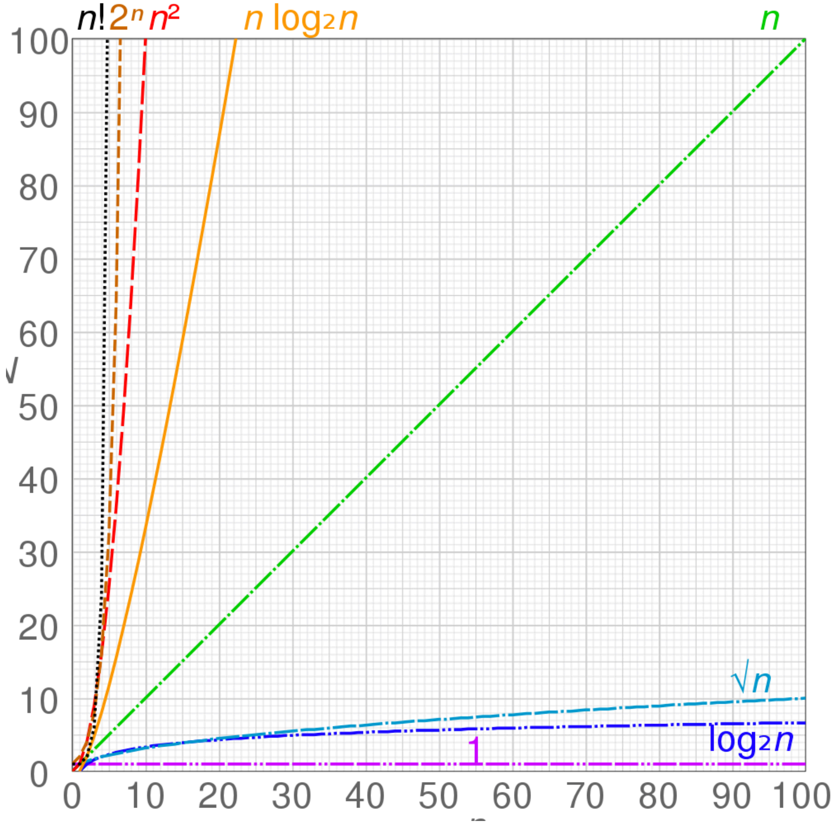
ST:



- Answering takes $O(f(n)\log n)$, where $f(n)$ is the time needed for processing a single node during an operation. Linear search above.

Square Root Complexity

- Algorithm w/ a O(√*n*) time complexity.
  - Poor man's logarithm.



Square Root Complexity

- A familiar problem: $\mathrm{sum}_q(a,b)$ and update/add.

PSA	$O(1)$	$O(n)$
BIT	$O(\log n)$	$O(\log n)$
ST	$O(\log n)$	$O(\log n)$

- Let's do it this way: $O(\sqrt{n})$ $O(1)$

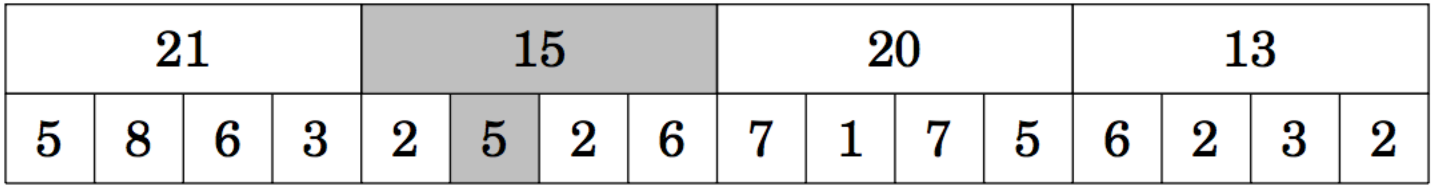
Square Root Complexity

- Divide the array into blocks of size $\sqrt{n}$ so that each block contains the sum of elements inside it.

21				17				20				13			
5	8	6	3	2	7	2	6	7	1	7	5	6	2	3	2

Square Root Complexity

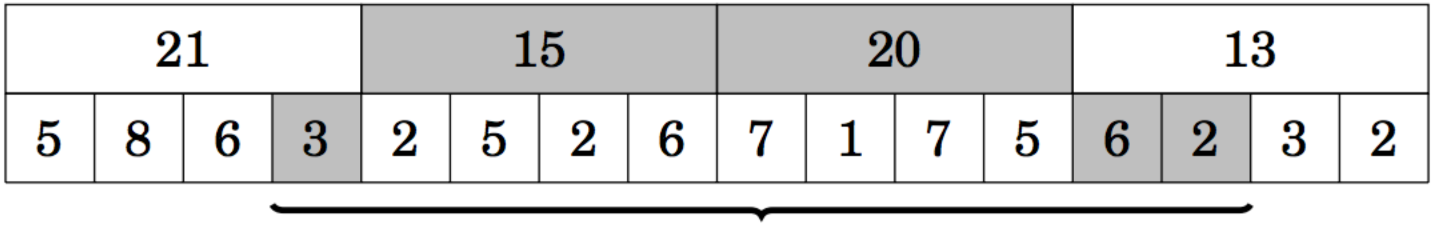
- Update the sum of a *single* block after each update, hence $O(1)$.



21				15				20				13			
5	8	6	3	2	5	2	6	7	1	7	5	6	2	3	2

Square Root Complexity

- For sum, divide the range into 3 parts s.t. the sum consists of values of single elements (3+6+2) and sums of blocks between them (15+20).



21				15				20				13			
5	8	6	3	2	5	2	6	7	1	7	5	6	2	3	2

- # of single elements is $O(\sqrt{n})$ //block size is $\sqrt{n}$.
- # of blocks is $O(\sqrt{n})$ //need $\sqrt{n}$ blocks to save n vals.
- Hence, range sum in $O(\sqrt{n})$ time. $\min_q(a,b)$ similar.

Square Root Complexity

- The purpose of the block size $\sqrt{n}$ is that it balances two things: the array is divided into $\sqrt{n}$ blocks, each of which contains $\sqrt{n}$ elements.
- In practice, divide into k blocks each of which contains n/k elements.
- Optiml parameter depends on the problem & input.
  - If an algo often goes through the blocks but rarely inspects single elements inside the blocks, it may be a good idea to increase block sizes: divide the array into $k < \sqrt{n}$ blocks, each of which contains $n/k > \sqrt{n}$ elments.

Optional Part

- Remaining slides are optional for Data Structure purposes.
- We dig more into square root complexity with examples from number theory.
- We also present a binary search algorithm for square root computation.

Square Root Complexity

- Some basic things related to prime numbers*.
  - Prime or not?
  - Euler's totient function.

* Prime number: natural number greater than 1 that has no divisors other than 1 and itself: 2, 3, 5, 7, ..

Square Root Complexity

- Is *n* prime?

Square Root Complexity

- Is n prime? iterate through all numbers from 2 to n-1. Return false if division successful. $O(n)$.

Square Root Complexity

- Is n prime? iterate through all numbers from 2 to $\sqrt{n}$. Return false if division successful. $O(\sqrt{n})$.
- If a number has a factor larger than $\sqrt{n}$, then it surely has a factor less than $\sqrt{n}$ (already checked); o/w their multiplication would be $>n$, contradction.
- Contradiction: $\sqrt{n} * \sqrt{n}+\underline{\epsilon} > n$.

$\underline{36}$

2 * 18   3 * 12   4 * 9   **6 * 6**

Square Root Complexity

- Is n prime? iterate through all numbers from 2 to $\sqrt{n}$. Return false if division successful. $O(\sqrt{n})$.
- A larger-than-$\sqrt{n}$ factor of n must be multiplied by a smaller factor that has already been checked.

$\underline{36}$

2 * 18

3 * 12

4 * 9

6 * 6

Square Root Complexity

- Is n prime? So, we will go up to $\sqrt{n}$. But 6 by 6 instead of 1 by 1. Still $O(\sqrt{n})$ but cool (6 times faster in practice).
- All primes (>3) are of the form $6k\pm1$. Why?

Square Root Complexity

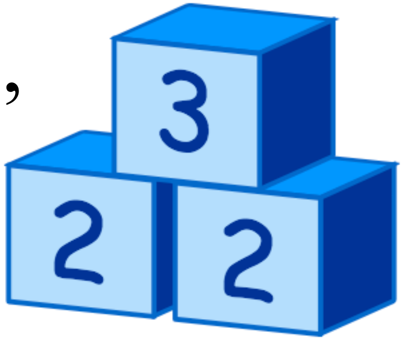
- Is n prime? So, we will go up to $\sqrt{n}$. But 6 by 6 instead of 1 by 1. Still $O(\sqrt{n})$ but cool (6 times faster in practice).
- All primes (>3) are of the form $6k\pm1$ ‘cos all numbers are of the form $6k+i$ for i=0..5.
- $6k+0$, $6k+2$, $6k+4$ are even (not prime). $6k+3$ divisible by 3 (not prime).
- So, $6k+1$ and $6k+5$ can be prime. Write as: $6k\pm1$.
- With this in mind, write the primality test code with increments of 6. ***see slide 89 for another cool pattern.***

Square Root Complexity

```
bool isPrime(int n){
 if(n<=1) ret false; if(n<=3) ret true;
 if(n%2==0 || n%3==0) ret false;
 for(i=5;i*i<=n;i+=6)
  if(n%i==0 || n%(i+2)==0) ret false;
 ret true; //6k-1  6k+1
}
```

Square Root Complexity

- Prime factorization: every number can be broken down into prime factors, i.e., prime numbers are the basic building blocks of all numbers: 12 = 2 * 2 * 3.



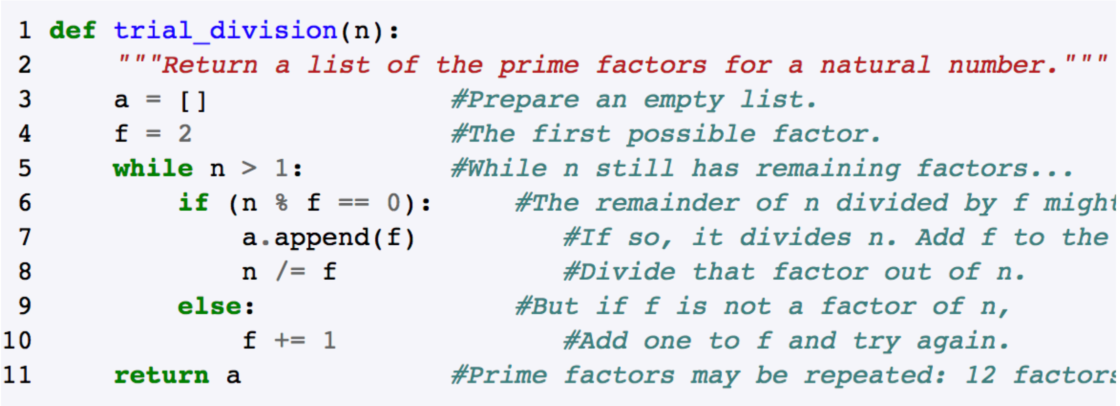
Square Root Complexity

- Prime factorization of n requires a search for prime factors in the range $[2,\sqrt{n}]$, hence $O(\sqrt{n})$*.
- There may be at most 1 prime factor in the range $[\sqrt{n},n]$ ‘cos o/w 2 factors’ multiplication would be $>n$, contradiction.

* We can find the unique prime factors in $O(\sqrt{n})$ by this search but cannot decide their multiplicity. That’s why prime factorization is very slow to solve for big numbers – foundation of cryptgraphy.

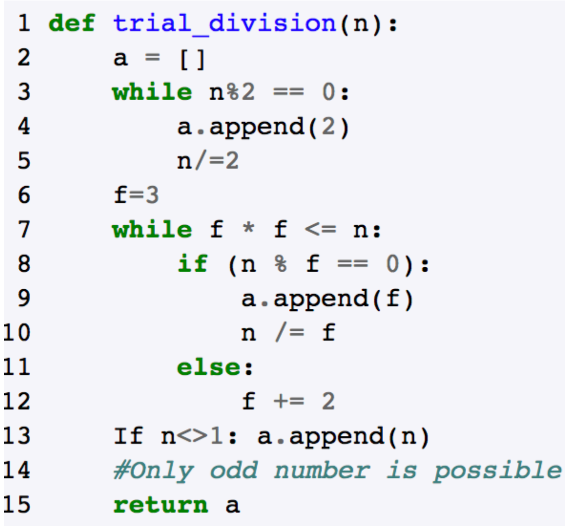
Square Root Complexity

- A simple prime factorization algo is Trial Division.



```python
def trial_division(n):
    """Return a list of the prime factors for a natural number."""
    a = []               #Prepare an empty list.
    f = 2                #The first possible factor.
    while n > 1:         #While n still has remaining factors...
        if (n % f == 0):     #The remainder of n divided by f might
            a.append(f)         #If so, it divides n. Add f to the
            n /= f              #Divide that factor out of n.
        else:                #But if f is not a factor of n,
            f += 1              #Add one to f and try again.
    return a             #Prime factors may be repeated: 12 factors
```

At least 2x more efficient (+=2):



```python
def trial_division(n):
    a = []
    while n%2 == 0:
        a.append(2)
        n/=2
    f=3
    while f * f <= n:
        if (n % f == 0):
            a.append(f)
            n /= f
        else:
            f += 2
    If n<>1: a.append(n)
    #Only odd number is possible
    return a
```

Some prime factorizations:

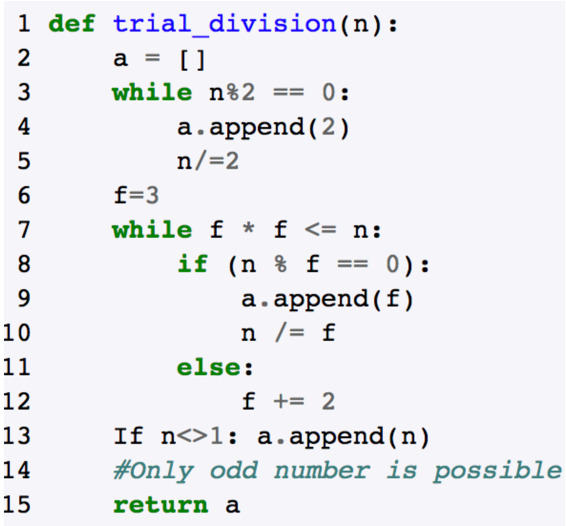
108	$2^2 \cdot 3^3$	128	2^7	148	$2^2 \cdot 37$	168	$2^3 \cdot 3 \cdot 7$	188	$2^2 \cdot 47$
109	**109**	129	$3 \cdot 43$	149	**149**	169	13^2	189	$3^3 \cdot 7$
110	$2 \cdot 5 \cdot 11$	130	$2 \cdot 5 \cdot 13$	150	$2 \cdot 3 \cdot 5^2$	170	$2 \cdot 5 \cdot 17$	190	$2 \cdot 5 \cdot 19$

Square Root Complexity

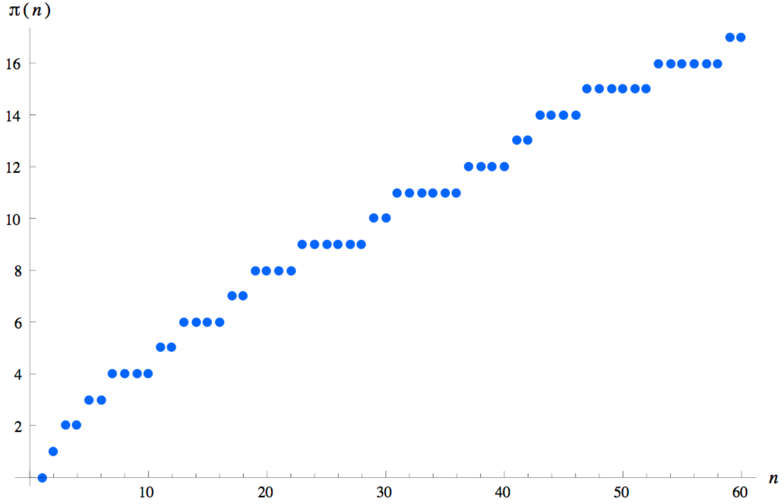
- For a base-2 *m*-digit number *n*, if we go from 3 to only $\sqrt{n}$, $\pi(2^{m/2}) \approx 2^{m/2} / ((m/2)\ln 2)$ divisions are required.

  $\pi(n)$: prime counting function,

  \# of primes less than *n*.



```
def trial_division(n):
    a = []
    while n%2 == 0:
        a.append(2)
        n/=2
    f=3
    while f * f <= n:
        if (n % f == 0):
            a.append(f)
            n /= f
        else:
            f += 2
    If n<>1: a.append(n)
    #Only odd number is possible
    return a
```

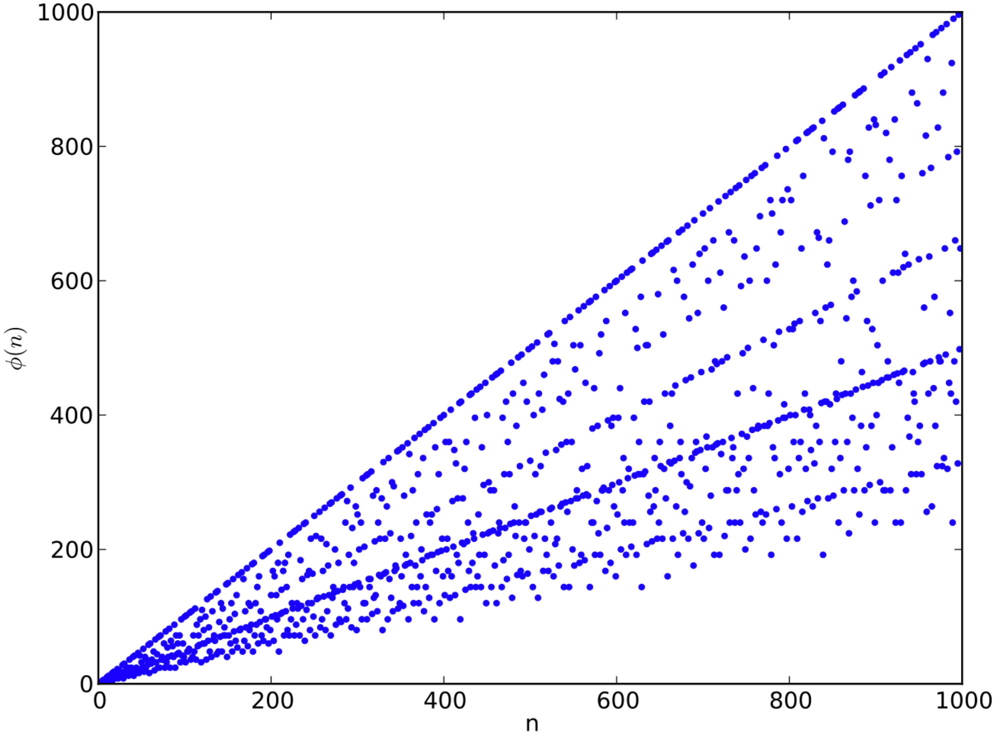


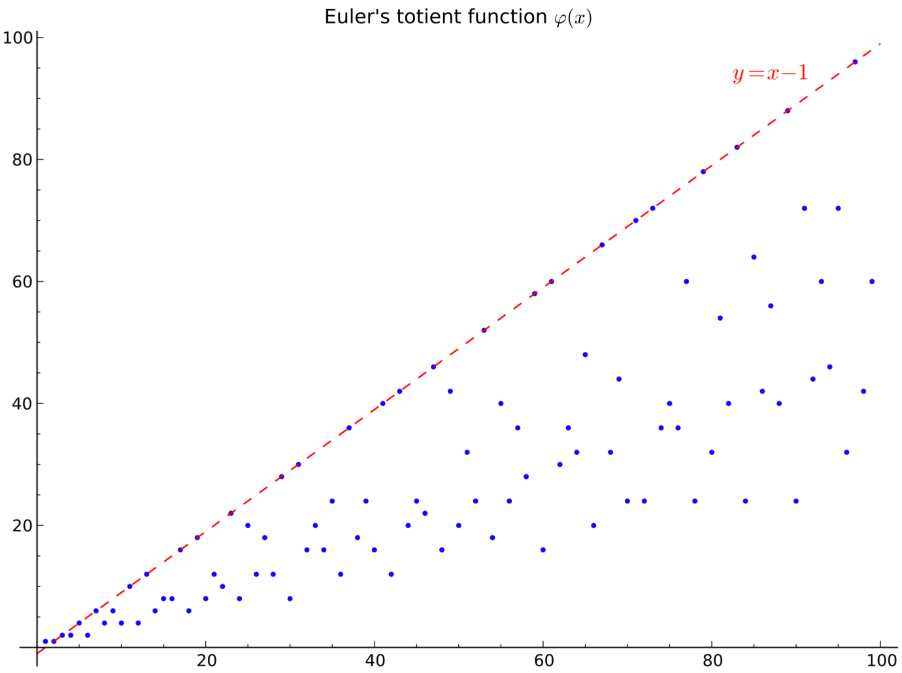
Square Root Complexity

- $\pi(2^{m/2})$ is exponential in m, the problem size.
  - Problem size is not n as we're dealing with 1 number whose value is n.

Square Root Complexity

- Euler’s totient function ϕ(*n*): # of +ve integers less than *n* that are relatively prime to *n*.





- ϕ(*n*) = *n*-1 if *n* is prime (top line). Makes sense!

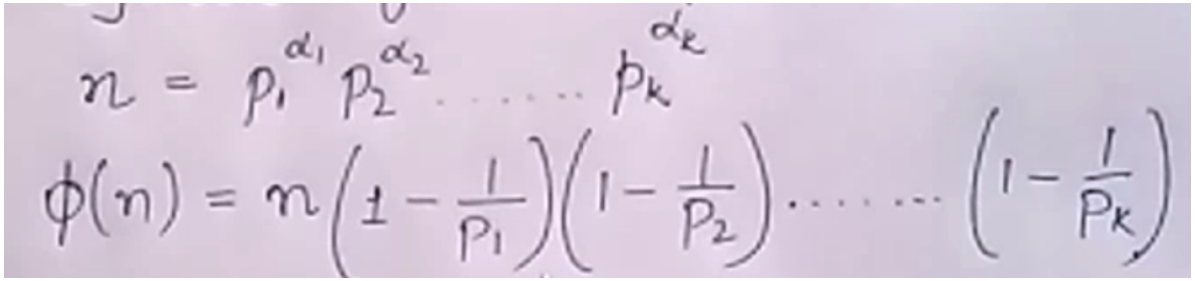
Square Root Complexity

- Euler's totient function ϕ(*n*): # of +ve integers less than *n* that are relatively prime to *n*.
- App: a regular *n*-gon can be constructed w/ ruler-and-compass technique if ϕ(*n*) is a power of 2.
- 6-gon creation:

http://ceng.metu.edu.tr/~ys/rulercompasshexagon-wiki.gif

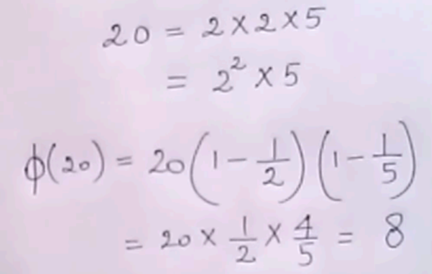
Square Root Complexity

- Euler's totient function ϕ(*n*): # of +ve integers less than *n* that are relatively prime to *n*.
- To compute ϕ(*n*), don't need the proper prime factorization since the exponents α_i aren't required.



$$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots\ldots p_k^{\alpha_k}$$

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\ldots\left(1 - \frac{1}{p_k}\right)$$



$$20 = 2 \times 2 \times 5$$
$$= 2^2 \times 5$$
$$\phi(20) = 20\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right)$$
$$= 20 \times \frac{1}{2} \times \frac{4}{5} = 8$$

- Hence, $O(\sqrt{n})$ time required (slide 82), no multiplicity (α_i) is required.

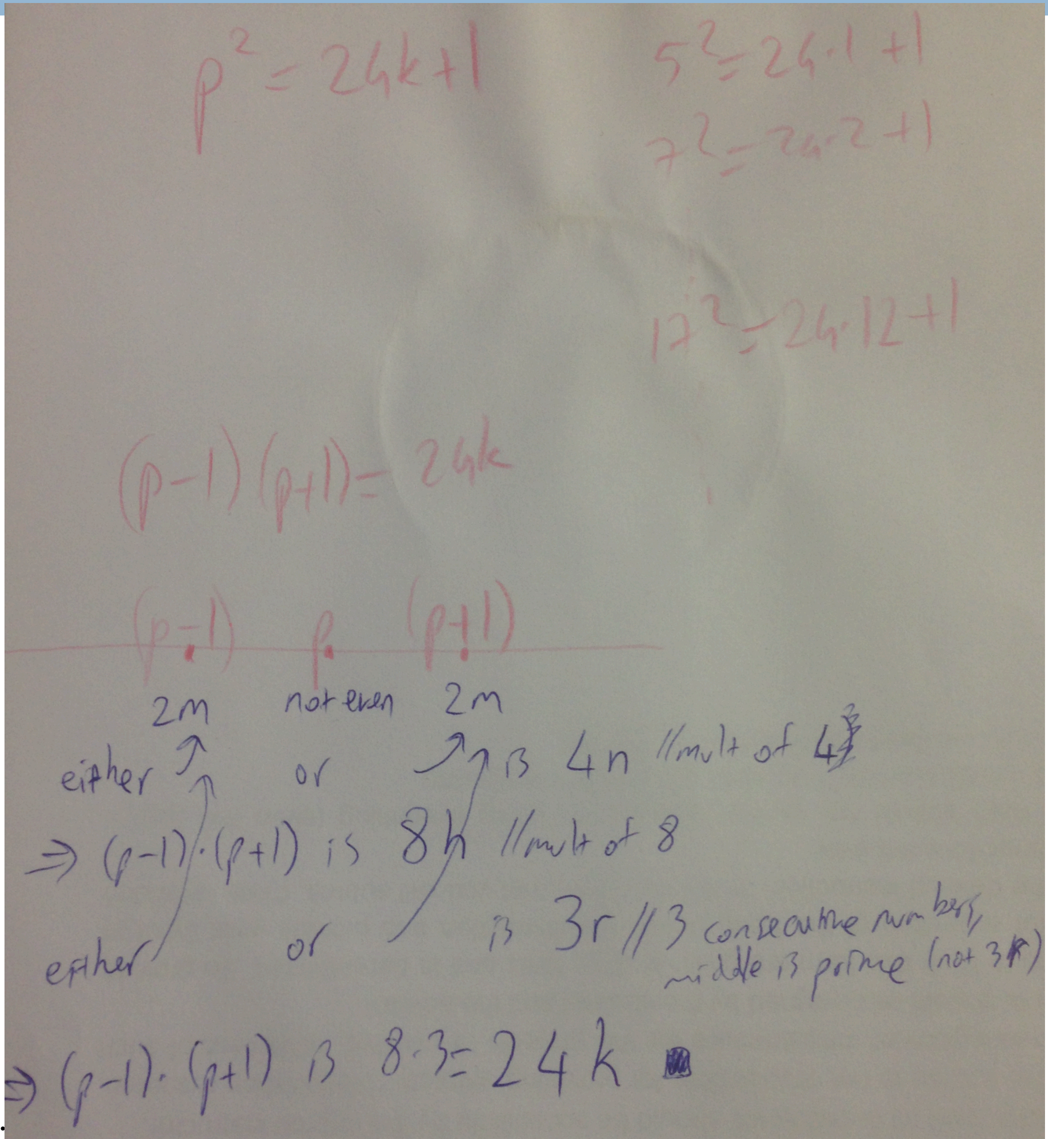
Square Root Complexity

- A cool pattern for primes: square of a prime is always one more than a multiple of 24.

Either p-1 or p+1 must be a multiple of 4: 4n.
Hence (p-1)(p+1) must be a multiple of 8: 8h.

Either (p-1) or (p+1) must be a multiple of 3: 3r.
Hence (p-1)(p+1) must be a multiple of 3: 3i.

Being multiples of 8 & 3, (p-1)(p+1) is multiple of 24.



$p^2 = 24k+1$

$5^2 = 24\cdot1+1$

$7^2 = 24\cdot2+1$

$17^2 = 24\cdot12+1$

$(p-1)(p+1) = 24k$

$(p-1)$ p $(p+1)$

2m not even 2m

either or is 4n //mult of 4

⇒ $(p-1)\cdot(p+1)$ is 8h //mult of 8

either or is 3r // 3 consecutive numbers, middle is prime (not 3k)

⇒ $(p-1)\cdot(p+1)$ is $8\cdot3 = 24k$ ■

Square Root Computation

- $\sqrt{n}$ computation algorithm in $O(\log n + p)$, where p is the # digits in fractional part: $\sqrt{10} = 3.162$ if p=3.

Square Root Computation

- $\sqrt{n}$ computation algorithm in $O(\log n + p)$.
- Integer part is found via binary search (n=10):

$\underset{s}{1}\ 2\ 3\ 4\ \underset{m}{5}\ 6\ 7\ 8\ 9\ \underset{e}{10}$ $5^2>10$ so go to left. //e=m-1.

$\underset{s}{1}\ \underset{m}{2}\ 3\ \underset{e}{4}\ 5\ 6\ 7\ 8\ 9\ 10$ $2^2<10$ so go to right. //s=m+1.

$1\ 2\ \underset{\underset{m}{s}}{3}\ \underset{e}{4}\ 5\ 6\ 7\ 8\ 9\ 10$ $3^2<10$ so go to right. //s=m+1.

Square Root Computation

- $\sqrt{n}$ computation algorithm in $O(\log n + p)$.
- Integer part is found via binary search (n=10):

1 2 3 4 5 6 7 8 9 10 $4^2>10$ so go to left. //$e=m$-1.
s
m
e

1 2 3 4 5 6 7 8 9 10 Break ‘cos $e<s$.
e s
3 vs. 4, 3 wins ‘cos $4^2>10$ and no recovery then.

- $O(\log n)$ time for the integer part.

Square Root Computation

- $\sqrt{n}$ computation algorithm in $O(\log n + p)$.
- Fractional part is found via linear search (p=3):

$3.?? = 10$

$3.1^2 < 10$

$3.2^2 > 10$ //stop,
keep 1.

Square Root Computation

- $\sqrt{n}$ computation algorithm in $O(\log n + p)$.
- Fractional part is found via linear search (p=2):

$3.?? = 10$

$3.1^2 < 10$

$3.2^2 > 10$ //stop,
keep 1.

$3.11^2 < 10$

$3.12^2 < 10$

:

$3.15^2 < 10$

$3.16^2 < 10$

$3.17^2 > 10$ //stop, keep 6.

Square Root Computation

- $\sqrt{n}$ computation algorithm in $O(\log n + p)$.
- Fractional part is found via linear search (p=3):
- At most 9 checks for each of p digits: $O(p)$.

- Overall, $O(\log n + p) \approx O(\log n)$ as p insignificant.